EXERCISES

1.1-1. Describe the outcome space for each of the following experiments:

(a) A student is selected at random from a statistics class, and the student's ACT score in mathematics is determined. HINT: ACT test scores in mathematics are integers between 1 and 36, inclusive.

(b) A candy bar with a 20.4-gram label weight is selected at random from a production line and is weighed.

(c) A coin is tossed three times, and the sequence of heads and tails is observed.

1.1-2. Describe the outcome space for each of the following experiments:

(a) Consider families that have three children each, and select one such family at random. Describe the sample space S in terms of their children as 3-tuples, agreeing, for example, that "gbb" would indicate that the youngest is a girl and the two oldest are boys.

(b) A rat is selected at random from a cage, and its sex is determined.

(c) A state selects a three-digit integer at random for one of its lottery games.

1.1-3. The State of Arizona generates a three-digit number at random six days a week for its daily lottery. The numbers are generated one digit at a time. Consider the following set of 50 three-digit numbers as 150 one-digit integers that were generated at random from May 1, 2000 to July 3, 2000:

951	728	818	922	850	835	003	406	203	603
011	217	803	776	397	019	785	185	632	245
945	929	508	849	516	729	306	305	278	100
089	860	918	124	675	220	728	751	786	609
076	320	732	911	913	556	367	897	540	979

Let X denote the outcome when a single digit is generated.

(a) With true random digits, what probability would you assign to each digit? That is, what is the p.m.f. of X?

(b) For the 150 observations, determine the relative frequencies of 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, respectively.

(c) Construct a probability histogram and a relative frequency histogram like those in Figure 1.1-2.

(d) A set of 32 three-digit numbers was generated from May 4, 1998 to June 9, 1998. Answer (b) and (c) for these data. What is true about the fit of the probability model to the data? (For a more complete discussion of these numbers, see "The Case of the Missing Lottery Number," by W. D. Kaigh, in *The College Mathematics Journal*, January, 2001, pages 15–19.) The numbers are as follows:

287	215	846	873	485	812	432	415	348	383
334	075	655	871	824	704	533	014	822	477
880	316	012	010	557	881	222	256	282	704
111	124								

1.1-4. Some ornithologists were interested in the clutch size of the common gallinule. They observed the number of eggs in each of 117 nests, yielding the following data:

7	5	13	7	7	8	9	9	9	8	8	9	9	7	7
5	9	7	7	4	9	8	8	10	9	7	8	8	8	7
9	7	7	10	8	7	9	7	10	8	9	7	11	10	9
9	4	8	6	8	9	9	9	8	8	5	8	8	9	9
14	10	8	9	9	9	8	7	9	7	9	10	10	7	6
11	7	7	6	9	7	7	6	8	9	4	6	9	8	9
7	9	9	9	9	8	8	8	9	9	9	8	10	9	9
8	5	7	8	7	6	7	7	7	6	5	9			

(a) Construct a frequency table for these data.
(b) Draw a histogram.
(c) What is the mode (the typical clutch size)?

1.1-5. Noticing that some of the gallinules in the last exercise had a second brood during the summer, the ornithologists became interested in comparing the clutch sizes for the second brood with those for the first brood. They were able to collect the following clutch sizes for the second brood:

4	4	6	5	5	9	6	5	6	9	7	9	4	8	6	5	9
8	8	7	6	8	8	9	10	9	9	7	4	6	8	7	5	

(a) Construct a frequency table for these data.
(b) Draw a histogram.
(c) Is there a typical clutch size for second broods?

1.1-6. Before buying enough cereal to obtain a set of four prizes (Example 1.1-4), a family decided to use simulation to estimate the number of boxes it would have to buy. Each member of the family rolled a four-sided die until he or she observed each face at least once. The family members repeated this exercise 100 times and recorded the numbers of rolls needed as follows:

8	6	6	4	13	11	19	13	4	15	8	13	5	8	9
16	9	5	12	6	4	8	6	6	6	11	6	5	10	5
6	5	8	4	5	14	5	7	5	7	5	8	4	9	8
9	13	8	14	5	24	4	5	6	7	5	4	8	7	6
11	4	5	6	5	10	10	4	5	14	10	15	6	9	8
7	10	14	10	8	8	10	7	9	10	8	10	9	7	5
12	7	16	6	5	11	4	11	5	5					

(a) Construct a frequency table for these data.
(b) Draw a histogram.

1.1-7. For a Texas lottery game, Cash Five, 5 integers are selected randomly out of the first 37 positive integers. The following table lists the numbers of odd integers out of each set of five integers selected in 100 consecutive drawings in 2007:

3	2	2	2	2	3	3	2	4	2	4	2	2	3	2	3	1	3	2	3
4	3	3	4	5	3	2	1	2	0	2	2	2	3	1	2	1	3	3	1
2	5	4	0	3	3	0	1	3	3	3	5	2	2	3	1	4	4	3	3
4	3	3	1	2	3	1	3	2	3	3	2	2	3	3	4	3	2	1	5
3	0	1	2	2	4	3	3	3	1	2	4	4	1	1	4	4	1	1	3

(a) Construct a frequency table for these data.
(b) Draw a histogram.

1.1-8. Consider a bowl containing 10 chips of the same size such that 2 are marked "one," 3 are marked "two," 3 are marked "three," and 2 are marked "four." Select a chip at random and read the number. Here $S = \{1, 2, 3, 4\}$.

(a) Assign a reasonable p.m.f. $f(x)$ to the outcome space.
(b) Simulate this experiment at least $n = 100$ times and find the relative frequency histogram $h(x)$. HINT: Here you can use a computer to perform the simulation; or simply use the table of random numbers (Table IX in the appendix), start at a random spot, and let an integer in the set $\{0, 1\} = 1$, in $\{2, 3, 4\} = 2$, in $\{5, 6, 7\} = 3$, in $\{8, 9\} = 4$.
(c) Plot $f(x)$ and $h(x)$ on the same graph.

1.1-9. Toss two coins at random and count the number of heads that appear "up." Here $S = \{0, 1, 2\}$. In Chapter 2, we discover that a reasonable probability model is given by the p.m.f. $f(0) = 1/4$, $f(1) = 1/2$, $f(2) = 1/4$. Repeat this experiment at least $n = 100$ times, and plot the resulting relative frequency histogram $h(x)$ on the same graph with $f(x)$.

1.1-10. Let the random variable X be the number of tosses of a coin needed to obtain the first head. Here $S = \{1, 2, 3, 4, \ldots\}$. In Chapter 2, we find that a reasonable probability model is given by the p.m.f. $f(x) = (1/2)^x$, $x \in S$. Do this experiment a large number of times, and compare the resulting relative frequency histogram $h(x)$ with $f(x)$.

1.1-11. In 1985, Al Bumbry of the Baltimore Orioles and Darrell Brown of the Minnesota Twins had the following numbers of hits (H) and official at bats (AB) on grass and artificial turf:

Playing Surface	Bumbry			Brown		
	AB	H	BA	AB	H	BA
Grass	295	77		92	18	
Artificial Turf	49	16		168	53	
Total	344	93		260	71	

(a) Find the batting averages BA (namely, H/AB) of each player on grass.
(b) Find the BA of each player on artificial turf.
(c) Find the season batting averages for the two players.
(d) Interpret your results.

1.1-12. In 1985, Kent Hrbek of the Minnesota Twins and Dion James of the Milwaukee Brewers had the following numbers of hits (H) and official at bats (AB) on grass and artificial turf:

Playing Surface	Hrbek			James		
	AB	H	BA	AB	H	BA
Grass	204	50		329	93	
Artificial Turf	355	124		58	21	
Total	559	174		387	114	

(a) Find the batting averages BA (namely, H/AB) of each player on grass.
(b) Find the BA of each player on artificial turf.
(c) Find the season batting averages for the two players.
(d) Interpret your results.

1.1-13. If we had a choice of two airlines, we would possibly choose the airline with the better "on-time performance." So consider Alaska Airlines and America West, using data reported by Arnold Barnett (see references):

Airline	Alaska Airlines	America West
Destination	Relative Frequency on Time	Relative Frequency on Time
Los Angeles	$\frac{497}{559}$	$\frac{694}{811}$
Phoenix	$\frac{221}{233}$	$\frac{4840}{5255}$
San Diego	$\frac{212}{232}$	$\frac{383}{448}$
San Francisco	$\frac{503}{605}$	$\frac{320}{449}$
Seattle	$\frac{1841}{2146}$	$\frac{201}{262}$
Five-City Total	$\frac{3274}{3775}$	$\frac{6438}{7225}$

(a) For each of the five cities listed, which airline has the better on-time performance?
(b) Combining the results, which airline has the better on-time performance?
(c) Interpret your results.

1.2 PROPERTIES OF PROBABILITY

In Section 1.1, the collection of all possible outcomes (the **universal set**) of a random experiment is denoted by S and is called the **outcome space**. Given an outcome space S, let A be a part of the collection of outcomes in S; that is, $A \subset S$. Then A is called an **event**. When the random experiment is performed and the outcome of the experiment is in A, we say that **event A has occurred**.

where $h = N(A)$ is the number of ways A can occur and $m = N(S)$ is the number of ways S can occur. Exercise 1.2-19 considers this assignment of probability in a more theoretical manner.

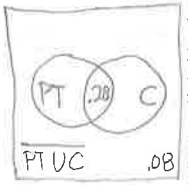
It should be emphasized that in order to assign the probability h/m to the event A, we must assume that each of the outcomes $e_1, e_2, \ldots, e_m$ has the same probability $1/m$. This assumption is then an important part of our probability model; if it is not realistic in an application, then the probability of the event A cannot be computed in this way. Actually, we have used this result in the simple case given in Example 1.2-3 because it seemed realistic to assume that each of the possible outcomes in $S = \{HH, HT, TH, TT\}$ had the same chance of being observed.

EXAMPLE 1.2-6 Let a card be drawn at random from an ordinary deck of 52 playing cards. Then the sample space S is the set of $m = 52$ different cards, and it is reasonable to assume that each of these cards has the same probability of selection, 1/52. Accordingly, if A is the set of outcomes that are kings, then $P(A) = 4/52 = 1/13$ because there are $h = 4$ kings in the deck. That is, 1/13 is the probability of drawing a card that is a king, provided that each of the 52 cards has the same probability of being drawn. ■

In Example 1.2-6, the computations are very easy because there is no difficulty in the determination of the appropriate values of h and m. However, instead of drawing only one card, suppose that 13 are taken at random and without replacement. Then we can think of each possible 13-card hand as being an outcome in a sample space, and it is reasonable to assume that each of these outcomes has the same probability. For example, to use the preceding method to assign the probability of a hand consisting of seven spades and six hearts, we must be able to count the number h of all such hands, as well as the number m of possible 13-card hands. In these more complicated situations, we need better methods of determining h and m. We discuss some of these counting techniques in Section 1.3.

EXERCISES

1.2-1. Of a group of patients having injuries, 28% visit both a physical therapist and a chiropractor and 8% visit neither. Say that the probability of visiting a physical therapist exceeds the probability of visiting a chiropractor by 16%. What is the probability of a randomly selected person from this group visiting a physical therapist? $1 - .08 = P(PT \cup C)$ = additivity



1.2-2. An insurance company looks at its auto insurance customers and finds that (a) all insure at least one car, (b) 85% insure more than one car, (c) 23% insure a sports car, and (d) 17% insure more than one car, including a sports car. Find the probability that a customer selected at random insures exactly one car and it is not a sports car.

$P(M) = .85$
$P(SP) = .23$

1.2-3. Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns 1/52 to each of the 52 outcomes. Let

$A = \{x : x \text{ is a jack, queen, or king}\}$,
$B = \{x : x \text{ is a 9, 10, or jack and } x \text{ is red}\}$,
$C = \{x : x \text{ is a club}\}$,
$D = \{x : x \text{ is a diamond, a heart, or a spade}\}$.

Find (a) $P(A)$, (b) $P(A \cap B)$, (c) $P(A \cup B)$, (d) $P(C \cup D)$, and (e) $P(C \cap D)$.

1.2-4. A coin is tossed four times, and the sequence of heads and tails is observed.

(a) List each of the 16 sequences in the sample space S.

(b) Let events A, B, C, and D be given by $A = \{\text{at least 3 heads}\}$, $B = \{\text{at most 2 heads}\}$, $C = \{\text{heads on the third toss}\}$, and $D = \{\text{1 head and 3 tails}\}$. If the probability set function assigns 1/16 to each outcome in the sample space, find (i) $P(A)$, (ii) $P(A \cap B)$, (iii) $P(B)$, (iv) $P(A \cap C)$, (v) $P(D)$, (vi) $P(A \cup C)$, and (vii) $P(B \cap D)$.

1.2-5. A field of beans is planted with three seeds per hill. For each hill of beans, let A_i be the event that i seeds germinate, $i = 0, 1, 2, 3$. Suppose that $P(A_0) = 1/64$, $P(A_1) = 9/64$, and $P(A_2) = 27/64$. Give the value of $P(A_3)$.

1.2-6. Consider the trial on which a 3 is first observed in successive rolls of a six-sided die. Let A be the event that 3 is observed on the first trial. Let B be the event that at least two trials are required to observe a 3. Assuming that each side has probability 1/6, find (a) $P(A)$, (b) $P(B)$, and (c) $P(A \cup B)$.

1.2-7. A fair eight-sided die is rolled once. Let $A = \{2, 4, 6, 8\}$, $B = \{3, 6\}$, $C = \{2, 5, 7\}$, and $D = \{1, 3, 5, 7\}$. Assume that each face has the same probability.

(a) Give the values of (i) $P(A)$, (ii) $P(B)$, (iii) $P(C)$, and (iv) $P(D)$.

(b) Give the values of (i) $P(A \cap B)$, (ii) $P(B \cap C)$, and (iii) $P(C \cap D)$.

(c) Give the values of (i) $P(A \cup B)$, (ii) $P(B \cup C)$, and (iii) $P(C \cup D)$, using Theorem 1.2-5.

1.2-8. If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, find (a) $P(A \cup B)$, (b) $P(A \cap B')$, and (c) $P(A' \cup B')$. For (b) use event decomp. method.

1.2-9. Given that $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$.

1.2-10. During a visit to a primary care physician's office, the probability of having neither lab work nor referral to a specialist is 0.21. Of those coming to that office, the probability of having lab work is 0.41 and the probability of having a referral is 0.53. What is the probability of having both lab work and a referral?

1.2-11. Roll a fair six-sided die three times. Let $A_1 = \{1 \text{ or } 2 \text{ on the first roll}\}$, $A_2 = \{3 \text{ or } 4 \text{ on the second roll}\}$, and $A_3 = \{5 \text{ or } 6 \text{ on the third roll}\}$. It is given that $P(A_i) = 1/3$, $i = 1, 2, 3$; $P(A_i \cap A_j) = (1/3)^2, i \neq j$; and $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.

(a) Use Theorem 1.2-6 to find $P(A_1 \cup A_2 \cup A_3)$.

(b) Show that $P(A_1 \cup A_2 \cup A_3) = 1 - (1 - 1/3)^3$.

1.2-12. Prove Theorem 1.2-6.

1.2-13. For each positive integer n, let $P(\{n\}) = (1/2)^n$. Consider the events $A = \{n : 1 \leq n \leq 10\}$, $B = \{n : 1 \leq n \leq 20\}$, and $C = \{n : 11 \leq n \leq 20\}$. Find (a) $P(A)$, (b) $P(B)$, (c) $P(A \cup B)$, (d) $P(A \cap B)$, (e) $P(C)$, and (f) $P(B')$.

1.2-14. Let x equal a number that is selected randomly from the closed interval from zero to one, [0, 1]. Use your intuition to assign values to

(a) $P(\{x : 0 \leq x \leq 1/3\})$.
(b) $P(\{x : 1/3 \leq x \leq 1\})$.
(c) $P(\{x : x = 1/3\})$.
(d) $P(\{x : 1/2 < x < 5\})$.

1.2-15. A typical roulette wheel used in a casino has 38 slots that are numbered $1, 2, 3, \ldots, 36, 0, 00$, respectively. The 0 and 00 slots are colored green. Half of the remaining slots are red and half are black. Also, half of the integers between 1 and 36 inclusive are odd, half are even, and 0 and 00 are defined to be neither odd nor even. A ball is rolled around the wheel and ends up in one of the slots; we assume that each slot has equal probability of 1/38, and we are interested in the number of the slot into which the ball falls.

(a) Define the sample space S.
(b) Let $A = \{0, 00\}$. Give the value of $P(A)$.
(c) Let $B = \{14, 15, 17, 18\}$. Give the value of $P(B)$.
(d) Let $D = \{x : x \text{ is odd}\}$. Give the value of $P(D)$.

1.2-16. The five numbers 1, 2, 3, 4, and 5 are written respectively on five disks of the same size and placed in a hat. Two disks are drawn without replacement from the hat, and the numbers written on them are observed.

(a) List the 10 possible outcomes of this experiment as unordered pairs of numbers.

(b) If each of the 10 outcomes has probability 1/10, assign a value to the probability that the sum of the two numbers drawn is (i) 3; (ii) between 6 and 8 inclusive.

1.2-17. Divide a line segment into two parts by selecting a point at random. Use your intuition to assign a probability to the event that the longer segment is at least two times longer than the shorter segment.

1.2-18. Let the interval $[-r, r]$ be the base of a semicircle. If a point is selected at random from this interval, assign a probability to the event that the length of the perpendicular segment from the point to the semicircle is less than $r/2$.

1.2-19. Let $S = A_1 \cup A_2 \cup \cdots \cup A_m$, where events $A_1, A_2, \ldots, A_m$ are mutually exclusive and exhaustive.

(a) If $P(A_1) = P(A_2) = \cdots = P(A_m)$, show that $P(A_i) = 1/m$, $i = 1, 2, \ldots, m$.

(b) If $A = A_1 \cup A_2 \cup \cdots \cup A_h$, where $h < m$, and (a) holds, prove that $P(A) = h/m$.

1.2-20. Let p_n, $n = 0, 1, 2, \ldots$, be the probability that an automobile policyholder will file for n claims in a five-year period. The actuary involved makes the assumption that $p_{n+1} = (1/4)p_n$. What is the probability that the holder will file two or more claims during this period?

EXERCISES

1.3-1. A boy found a bicycle lock for which the combination was unknown. The correct combination is a four-digit number, $d_1 d_2 d_3 d_4$, where d_i, $i = 1, 2, 3, 4$, is selected from 1, 2, 3, 4, 5, 6, 7, and 8. How many different lock combinations are possible with such a lock?

1.3-2. How many different orchid displays in a line are possible using four orchids of different colors if exactly three orchids are used?

1.3-3. How many different license plates are possible if a state uses

(a) Two letters followed by a four-digit integer (leading zeros are permissible and the letters and digits can be repeated)?

(b) Three letters followed by a three-digit integer? (In practice, it is possible that certain "spellings" are ruled out.)

1.3-4. In designing an experiment, the researcher can often choose many different levels of the various factors in order to try to find the best combination at which to operate. As an illustration, suppose the researcher is studying a certain chemical reaction and can choose four levels of temperature, five different pressures, and two different catalysts.

(a) To consider all possible combinations, how many experiments would need to be conducted?

(b) Often in preliminary experimentation, each factor is restricted to two levels. With the three factors noted, how many experiments would need to be run to cover all possible combinations with each of the three factors at two levels? (NOTE: This is often called a 2^3 design.)

1.3-5. How many four-letter code words are possible using the letters in HOPE if

(a) The letters may not be repeated?

(b) The letters may be repeated?

1.3-6. The "eating club" is hosting a make-your-own sundae at which the following are provided:

Ice Cream Flavors	Toppings
Chocolate	Caramel
Cookies-n-cream	Hot fudge
Strawberry	Marshmallow
Vanilla	M&M's
	Nuts
	Strawberries

(a) How many sundaes are possible using one flavor of ice cream and three different toppings?

(b) How many sundaes are possible using one flavor of ice cream and from zero to six toppings?

(c) How many different combinations of flavors of three scoops of ice cream are possible if it is permissible to make all three scoops the same flavor?

1.3-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

(a) $6, 7, 8, 9$.

(b) $6, 7, 8, 8$.

(c) $7, 7, 8, 8$.

(d) $7, 8, 8, 8$.

1.3-8. From a collection of nine paintings, four are to be selected to hang side by side on a gallery wall in positions 1, 2, 3, and 4. In how many ways can this be done?

1.3-9. Some albatrosses return to the world's only mainland colony of royal albatrosses, on Otago Peninsula near Dunedin, New Zealand, every two years to nest and raise their young. In order to learn more about the albatross, colored plastic bands are placed on their legs so that they can be identified from a distance. Suppose that three bands are placed on one leg, with the color of the band selected from the colors red, yellow, green, white, and blue. Find the number of different color codes that are possible for banding an albatross if

(a) The three bands are of different colors.
(b) Repeated colors are permissible.

1.3-10. Hope and Calvin play volleyball until one team wins three games. Using **H** for a Hope victory and **C** for a Calvin victory, and considering the possible orderings for the winning team, in how many ways could the volleyball match end?

1.3-11. The World Series in baseball continues until either the American League team or the National League team wins four games. How many different orders are possible (e.g., *ANNAAA* means the American League team wins in six games) if the series goes

(a) Four games?
(b) Five games?
(c) Six games?
(d) Seven games?

1.3-12. How many different varieties of pizza can be made if you have the following choices: small, medium, or large size; thin 'n crispy, hand-tossed, or pan crust; and 12 toppings (cheese is automatic), from which you may select from 0 to 12?

1.3-13. A cafe lets you order a deli sandwich your way. There are six choices for bread, four choices for meat, four choices for cheese, and 12 different garnishes (condiments). How many different sandwich possibilities are there if you choose

(a) One bread, one meat, and one cheese?
(b) One bread, one meat, one cheese, and from 0 to 12 garnishes?
(c) One bread; 0, 1, or 2 meats; 0, 1, or 2 cheeses; and from 0 to 12 garnishes?

1.3-14. Pascal's triangle gives a method for calculating the binomial coefficients; it begins as follows:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

The nth row of this triangle gives the coefficients for $(a+b)^{n-1}$. To find an entry in the table other than a 1 on the boundary, add the two nearest numbers in the row directly above. The equation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1},$$

called **Pascal's equation**, explains why Pascal's triangle works. Prove that this equation is correct.

1.3-15. Three students (S) and six faculty members (F) are on a panel discussing a new college policy.

(a) In how many different ways can the nine participants be lined up at a table in the front of the auditorium?
(b) How many lineups are possible, considering only the labels S and F?
(c) For each of the nine participants, you are to decide whether the participant did a good job or a poor job stating his or her opinion of the new policy; that is, give each of the nine participants a grade of G or P. How many different "scorecards" are possible?

1.3-16. Prove:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0 \quad \text{and} \quad \sum_{r=0}^{n}\binom{n}{r} = 2^n.$$

Hint: Consider $(1-1)^n$ and $(1+1)^n$, or use Pascal's equation and proof by induction.

1.3-17. A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:

(a) Four of a kind (4 cards of equal face value and 1 card of a different value).
(b) Full house (one pair and one triple of cards with equal face value).
(c) Three of a kind (three equal face values plus 2 cards of different values).
(d) Two pairs (two pairs of equal face value plus 1 card of a different value).
(e) One pair (one pair of equal face value plus 3 cards of different values).

1.3-18. Prove Equation 1.3-2. Hint: First select n_1 positions in $\binom{n}{n_1}$ ways. Then select n_2 from the remaining $n - n_1$ positions in $\binom{n-n_1}{n_2}$ ways, and so on. Finally, use the multiplication rule.

1.3-19. There are three teams in a cross-country race. Team A has five runners, team B has six runners, and team C has seven runners. In how many ways can the runners cross the finish line if we are interested only in the team for which they

run? That is, what is the number of distinguishable permutations of five A's, six B's, and seven C's? (Note that, for scoring purposes, only the scores of the first five runners for each team count.)

1.3-20. A box of candy hearts contains 52 hearts, of which 19 are white, 10 are tan, 7 are pink, 3 are purple, 5 are yellow, 2 are orange, and 6 are green. If you select 9 pieces of candy randomly from the box, without replacement, give the probability that

(a) Three of the hearts are white.
(b) Three are white, 2 are tan, 1 is pink, 1 is yellow, and 2 are green.

1.3-21. An office furniture manufacturer that makes modular storage files offers its customers two choices for the base and four choices for the top, and the modular storage files come in five different heights. The customer may choose any combination of the five different-sized modules so that the finished file has a base, a top, and one, two, three, four, five, or six storage modules.

(a) How many choices does the customer have if the completed file has four storage modules, a top, and a base? The order in which the four modules are stacked is irrelevant.
(b) In its advertising, the manufacturer would like to use the number of different files that are possible—selecting one of the two bases, one of the four tops, and then either one, two, three, four, five, or six storage modules. The manufacturer may select any combination of the five different sizes, with the order of stacking irrelevant. What is the number of possibilities?

1.3-22. A bag of 36 dum-dum pops (suckers) contains up to 10 flavors. That is, there are from 0 to 36 suckers of each of 10 flavors in the bag. How many different flavor combinations are possible?

1.4 CONDITIONAL PROBABILITY

We introduce the idea of conditional probability by means of an example.

EXAMPLE 1.4-1 Suppose that we are given 20 tulip bulbs that are similar in appearance and told that 8 will bloom early, 12 will bloom late, 13 will be red, and 7 will be yellow, in accordance with the various combinations listed in Table 1.4-1. If one bulb is selected at random, the probability that it will produce a red tulip (R) is given by $P(R) = 13/20$, under the assumption that each bulb is "equally likely." Suppose, however, that close examination of the bulb will reveal whether it will bloom early (E) or late (L). If we consider an outcome only if it results in a tulip bulb that will bloom early, only eight outcomes in the sample space are now of interest. Thus, under this limitation, it is natural to assign the probability 5/8 to R; that is, $P(R \mid E) = 5/8$, where $P(R \mid E)$ is read as the probability of R given that E has occurred. Note that

$$P(R \mid E) = \frac{5}{8} = \frac{N(R \cap E)}{N(E)} = \frac{N(R \cap E)/20}{N(E)/20} = \frac{P(R \cap E)}{P(E)},$$

where $N(R \cap E)$ and $N(E)$ are the numbers of outcomes in events $R \cap E$ and E, respectively. ■

TABLE 1.4-1: Tulip Combinations

	Early (E)	Late (L)	Totals
Red (R)	5	8	13
Yellow(Y)	3	4	7
Totals	8	12	20

EXERCISES

1.4-1. A common test for AIDS is called the ELISA (enzyme-linked immunosorbent assay) test. Among 1 million people who are given the ELISA test, we can expect results similar to those given in the following table:

	B_1: Carry AIDS Virus	B_2: Do Not Carry AIDS Virus	Totals
A_1: Test Positive	4,885	73,630	78,515
A_2: Test Negative	115	921,370	921,485
Totals	5,000	995,000	1,000,000

If one of these 1 million people is selected randomly, find the following probabilities: (a) $P(B_1)$, (b) $P(A_1)$, (c) $P(A_1 \mid B_2)$, (d) $P(B_1 \mid A_1)$. (e) In words, what do parts (c) and (d) say?

1.4-2. The following table classifies 1456 people by their gender and by whether or not they favor a gun law.

	Male (S_1)	Female (S_2)	Totals
Favor (A_1)	392	649	1041
Oppose (A_2)	241	174	415
Totals	633	823	1456

Compute the following probabilities if one of these 1456 persons is selected randomly: (a) $P(A_1)$, (b) $P(A_1 \mid S_1)$, (c) $P(A_1 \mid S_2)$. (d) Interpret your answers to parts (b) and (c).

1.4-3. Let A_1 and A_2 be the events that a person is left-eye dominant or right-eye dominant, respectively. When a person folds his or her hands, let B_1 and B_2 be the events that the left thumb and right thumb, respectively, are on top. A survey in one statistics class yielded the following table:

	B_1	B_2	Totals
A_1	5	7	12
A_2	14	9	23
Totals	19	16	35

If a student is selected randomly, find the following probabilities: (a) $P(A_1 \cap B_1)$, (b) $P(A_1 \cup B_1)$, (c) $P(A_1 \mid B_1)$, (d) $P(B_2 \mid A_2)$. (e) If the students had their hands folded and you hoped to select a right-eye-dominant student, would you select a "right thumb on top" or a "left thumb on top" student? Why?

1.4-4. Two cards are drawn successively and without replacement from an ordinary deck of playing cards. Compute the probability of drawing

(a) Two hearts.
(b) A heart on the first draw and a club on the second draw.
(c) A heart on the first draw and an ace on the second draw.

HINT: In part (c), note that a heart can be drawn by getting the ace of hearts or one of the other 12 hearts.

1.4-5. Suppose that $P(A) = 0.7$, $P(B) = 0.5$, and $P([A \cup B]') = 0.1$.

(a) Find $P(A \cap B)$.
(b) Calculate $P(A \mid B)$.
(c) Calculate $P(B \mid A)$.

1.4-6. A hand of 13 cards is to be dealt at random and without replacement from an ordinary deck of playing cards. Find the conditional probability that there are at least three kings in the hand, given that the hand contains at least two kings.

1.4-7. Suppose that the genes for eye color for a certain male fruit fly are (R, W) and the genes for eye color for the mating female fruit fly are (R, W), where R and W represent red and white, respectively. Their offspring receive one gene for eye color from each parent.

(a) Define the sample space of the genes for eye color for the offspring.
(b) Assume that each of the four possible outcomes has equal probability. If an offspring ends up with either two red genes or one red and one white gene for eye color, its eyes will look red. Given that an offspring's eyes look red, what is the conditional probability that it has two red genes for eye color?

1.4-8. A researcher finds that, of 982 men who died in 2002, 221 died from some heart disease. Also, of the 982 men, 334 had at least one parent who had some heart disease. Of the latter 334 men, 111 died from some heart disease. A man is selected from the group of 982. Given that neither of his parents had some heart disease, find the conditional probability that this man died of some heart disease.

1.4-9. An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

1.4-10. An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting a single ball at random from the urn without replacement. The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls.

(a) If you draw first, find the probability that you win the game on your second draw.
(b) If you draw first, find the probability that your opponent wins the game on his second draw.
(c) If you draw first, what is the probability that you win? HINT: You could win on your second, third, fourth, ..., or tenth draw, but not on your first.
(d) Would you prefer to draw first or second? Why?

1.4-11. In a string of 12 Christmas tree light bulbs, 3 are defective. The bulbs are selected at random and tested, one at a time, until the third defective bulb is found. Compute the probability that the third defective bulb is the

(a) Third bulb tested.
(b) Fifth bulb tested.
(c) Tenth bulb tested.

1.4-12. A small grocery store had 10 cartons of milk, 2 of which were sour. If you are going to buy the sixth carton of milk sold that day at random, compute the probability of selecting a carton of sour milk.

1.4-13. In the gambling game "craps," a pair of dice is rolled and the outcome of the experiment is the sum of the points on the up sides of the six-sided dice. The bettor wins on the first roll if the sum is 7 or 11. The bettor loses on the first roll if the sum is 2, 3, or 12. If the sum is 4, 5, 6, 8, 9, or 10, that number is called the bettor's "point." Once the point is established, the rule is as follows: If the bettor rolls a 7 before the point, the bettor loses; but if the point is rolled before a 7, the bettor wins.

(a) List the 36 outcomes in the sample space for the roll of a pair of dice. Assume that each of them has a probability of 1/36.
(b) Find the probability that the bettor wins on the first roll. That is, find the probability of rolling a 7 or 11, $P(7 \text{ or } 11)$.
(c) Given that 8 is the outcome on the first roll, find the probability that the bettor now rolls the point 8 before rolling a 7 and thus wins. Note that at this stage in the game the only outcomes of interest are 7 and 8. Thus, find $P(8 \mid 7 \text{ or } 8)$.
(d) The probability that a bettor rolls an 8 on the first roll and then wins is given by $P(8)P(8 \mid 7 \text{ or } 8)$. Show that this probability is $(5/36)(5/11)$.
(e) Show that the total probability that a bettor wins in the game of craps is 0.49293. HINT: Note that the bettor can win in one of several mutually exclusive ways: by rolling a 7 or an 11 on the first roll or by establishing one of the points 4, 5, 6, 8, 9, or 10 on the first roll and then obtaining that point on successive rolls before a 7 comes up.

1.4-14. A single card is drawn at random from each of six well-shuffled decks of playing cards. Let A be the event that all six cards drawn are different.

(a) Find $P(A)$.
(b) Find the probability that at least two of the drawn cards match.

1.4-15. Consider the birthdays of the students in a class of size r. Assume that the year consists of 365 days.

(a) How many different ordered samples of birthdays are possible (r in sample) allowing repetitions (with replacement)?
(b) The same as part (a), except requiring that all the students have different birthdays (without replacement)?
(c) If we can assume that each ordered outcome in part (a) has the same probability, what is the probability that at least two students have the same birthday?
(d) For what value of r is the probability in part (c) about equal to 1/2? Is this number surprisingly small? HINT: Use a calculator or computer to find r.

1.4-16. You are a member of a class of 18 students. A bowl contains 18 chips: 1 blue and 17 red. Each student is to take 1 chip from the bowl without replacement. The student who draws the blue chip is guaranteed an A for the course.

(a) If you have a choice of drawing first, fifth, or last, which position would you choose? Justify your choice on the basis of probability.
(b) Suppose the bowl contains 2 blue and 16 red chips. What position would you now choose?

1.4-17. A drawer contains four black, six brown, and eight olive socks. Two socks are selected at random from the drawer.

(a) Compute the probability that both socks are the same color.

(b) Compute the probability that both socks are olive if it is known that they are the same color.

1.4-18. Bowl A contains three red and two white chips, and bowl B contains four red and three white chips. A chip is drawn at random from bowl A and transferred to bowl B. Compute the probability of then drawing a red chip from bowl B.

1.4-19. An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if ball numbered m is the mth ball selected. Let the event A_i denote a match on the ith draw, $i = 1, 2, 3, 4$.

(a) Show that $P(A_i) = \dfrac{3!}{4!}$.

(b) Show that $P(A_i \cap A_j) = \dfrac{2!}{4!}$.

(c) Show that $P(A_i \cap A_j \cap A_k) = \dfrac{1!}{4!}$.

(d) Show that the probability of at least one match is

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}.$$

(e) Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!}$$

$$= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).$$

(f) What is the limit of this probability as n increases without bound?

1.4-20. Paper is often tested for "burst strength" and "tear strength." Say we classify these strengths as low, middle, and high. Then, after examining 100 pieces of paper, we find the following:

	Burst Strength		
Tear Strength	A_1 (low)	A_2 (middle)	A_3 (high)
B_1 (low)	7	11	13
B_2 (middle)	11	21	9
B_3 (high)	12	9	7

If we select one of the pieces at random, what are the probabilities that it has the following characteristics:

(a) A_1,

(b) $A_3 \cap B_2$,

(c) $A_2 \cup B_3$,

(d) A_1, given that it is B_2,

(e) B_1, given that it is A_3?

1.4-21. An urn contains eight red and seven blue balls. A second urn contains an unknown number of red balls and nine blue balls. A ball is drawn from each urn at random, and the probability of getting two balls of the same color is 151/300. How many red balls are in the second urn?

EXERCISES

1.5-1. Let A and B be independent events with $P(A) = 0.7$ and $P(B) = 0.2$. Compute (a) $P(A \cap B)$, (b) $P(A \cup B)$, and (c) $P(A' \cup B')$.

1.5-2. Let $P(A) = 0.3$ and $P(B) = 0.6$.

(a) Find $P(A \cup B)$ when A and B are independent.

(b) Find $P(A \mid B)$ when A and B are mutually exclusive.

1.5-3. Let A and B be independent events with $P(A) = 1/4$ and $P(B) = 2/3$. Compute (a) $P(A \cap B)$, (b) $P(A \cap B')$, (c) $P(A' \cap B')$, (d) $P[(A \cup B)']$, and (e) $P(A' \cap B)$.

1.5-4. Prove parts (b) and (c) of Theorem 1.5-1.

1.5-5. If $P(A) = 0.8$, $P(B) = 0.5$, and $P(A \cup B) = 0.9$, are A and B independent events? Why or why not?

1.5-6. Show that if A, B, and C are mutually independent, then the following pairs of events are independent: A and $(B \cap C)$, A and $(B \cup C)$, A' and $(B \cap C')$. Show also that A', B' and C' are mutually independent.

1.5-7. Each of three football players will attempt to kick a field goal from the 25-yard line. Let A_i denote the event that the field goal is made by player i, $i = 1, 2, 3$. Assume that A_1, A_2, A_3 are mutually independent and that $P(A_1) = 0.5, P(A_2) = 0.7, P(A_3) = 0.6$.

(a) Compute the probability that exactly one player is successful.

(b) Compute the probability that exactly two players make a field goal (i.e., one misses).

1.5-8. Die A has orange on one face and blue on five faces, Die B has orange on two faces and blue on four faces, Die C has orange on three faces and blue on three faces. All are unbiased dice. If the three dice are rolled, find the probability that exactly two of the three dice come up orange.

1.5-9. Suppose that A, B, and C are mutually independent events and that $P(A) = 0.5$, $P(B) = 0.8$, and $P(C) = 0.9$. Find the probabilities that (a) all three events occur, (b) exactly two of the three events occur, and (c) none of the events occur.

1.5-10. Let D_1, D_2, D_3 be three four-sided dice whose sides have been labeled as follows:

$$D_1 : 0\,3\,3\,3 \qquad D_2 : 2\,2\,2\,5 \qquad D_3 : 1\,1\,4\,6$$

The three dice are rolled at random. Let A, B, and C be the events that the outcome on die D_1 is larger than the outcome on D_2, the outcome on D_2 is larger than the outcome on D_3, and the outcome on D_3 is larger than the outcome on D_1, respectively. Show that (a) $P(A) = 9/16$, (b) $P(B) = 9/16$, and (c) $P(C) = 10/16$. Do you find it interesting that each of the probabilities that D_1 "beats" D_2, D_2 "beats" D_3, and D_3 "beats" D_1 is greater than 1/2? Thus, it is difficult to determine the "best" die.

1.5-11. Let A and B be two events.

(a) If the events A and B are mutually exclusive, are A and B always independent? If the answer is no, can they ever be independent? Explain.

(b) If $A \subset B$, can A and B ever be independent events? Explain.

1.5-12. Flip an unbiased coin five independent times. Compute the probability of

(a) *HHTHT*.

(b) *THHHT*.

(c) *HTHTH*.

(d) Three heads occurring in the five trials.

1.5-13. An urn contains two red balls and four white balls. Sample successively five times at random and with replacement, so that the trials are independent. Compute the probability of each of the two sequences *WWRWR* and *RWWWR*.

1.5-14. In Example 1.5-5, suppose that the probability of failure of a component is $p = 0.4$. Find the probability that the system does not fail if the number of redundant components is

(a) 3.

(b) 8.

1.5-15. An urn contains 10 red and 10 white balls. The balls are drawn from the urn at random, one at a time. Find the probability that the fourth white ball is the sixth ball drawn if the sampling is done

(a) With replacement.

(b) Without replacement.

(c) In the World Series, the American League (red) and National League (white) teams play until one team wins four games. Do you think that the urn model presented in this exercise could be used to describe the probabilities of a 4-, 5-, 6-, or 7-game series? If your answer is yes, would you choose sampling with or without replacement in your model? (For your information, the numbers of 4-, 5-, 6-, and 7-game series, up to and including 2008, were 20, 23, 21, and 36, respectively. The World Series was canceled in 1994, and in 1903 and 1919–1921 winners had to take five out of nine games. Three of those series went eight games and one went seven.)

1.5-16. An urn contains five balls, one marked WIN and four marked LOSE. You and another player take turns selecting a ball at random from the urn, one at a time. The first person to select the WIN ball is the winner. If you draw first, find the probability that you will win if the sampling is done

(a) With replacement.

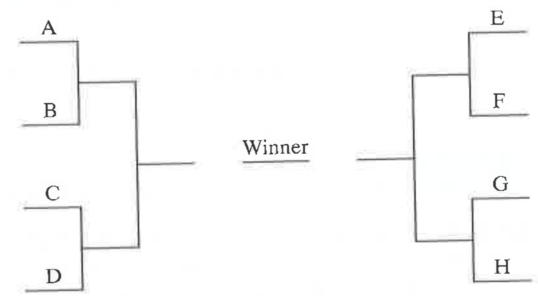
(b) Without replacement.

1.5-17. Each of the 12 students in a class is given a fair 12-sided die. In addition, each student is numbered from 1 to 12.

(a) If the students roll their dice, what is the probability that there is at least one "match" (e.g., student 4 rolls a 4)?

(b) If you are a member of this class, what is the probability that at least one of the other 11 students rolls the same number as you do?

1.5-18. An eight-team single-elimination tournament is set up as follows:



For example, eight students (called A–H) set up a tournament among themselves. The top-listed student in each bracket calls heads or tails when his or her opponent flips a coin. If the call is correct, the student moves on to the next bracket.

(a) How many coin flips are required to determine the tournament winner?

(b) What is the probability that you can predict all of the winners?

(c) In NCAA Division I basketball, after a "play-in" game between the 64th and 65th seeds, 64 teams participate in a single-elimination tournament to determine the national champion. Considering only the remaining 64 teams, how many games are required to determine the national champion?

(d) Assume that for any given game, either team has an equal chance of winning. (That is probably not true.) On page 43 of the March 22, 1999, issue, *Time*, claimed that the "mathematical odds of predicting all 63 NCAA games correctly is 1 in 75 million." Do you agree with this statement? If not, why not?

1.5-19. Extend Example 1.5-6 to an n-sided die. That is, suppose that a fair n-sided die is rolled n independent times. A match occurs if side i is observed on the ith trial, $i = 1, 2, \ldots, n$.

(a) Show that the probability of at least one match is

$$1 - \left(\frac{n-1}{n}\right)^n = 1 - \left(1 - \frac{1}{n}\right)^n.$$

(b) Find the limit of this probability as n increases without bound.

1.5-20. An urn contains n balls numbered from 1 through n. A sample of n balls is selected at random from the urn one at a time. A match occurs if ball i is selected on the ith draw.

(a) For $n = 1$ to 15, find the probability of at least one match if the sampling is done (i) with replacement (see Exercise 1.5-19), (ii) without replacement (see Exercise 1.4-19).

(b) How much does n affect these probabilities?

(c) How does sampling with and without replacement affect these probabilities?

(d) Illustrate these probabilities empirically, either performing the experiments physically or simulating them on a computer.

1.6 BAYES'S THEOREM

We begin this section by illustrating Bayes's theorem with an example.

EXAMPLE 1.6-1 Bowl B_1 contains two red and four white chips, bowl B_2 contains one red and two white chips, and bowl B_3 contains five red and four white chips. Say that the probabilities for selecting the bowls are not the same but are given by $P(B_1) = 1/3$, $P(B_2) = 1/6$, and $P(B_3) = 1/2$, where B_1, B_2, and B_3 are the events that bowls B_1, B_2, and B_3 are respectively chosen. The experiment consists of selecting a bowl with these probabilities and then drawing a chip at random from that bowl. Let us compute the probability of event R, drawing a red chip—say, $P(R)$. Note that $P(R)$ is dependent first of all on which bowl is selected and then on the probability of drawing a red chip from the selected bowl. That is, the event R is the union of the mutually exclusive events $B_1 \cap R$, $B_2 \cap R$, and $B_3 \cap R$. Thus,

$$\begin{aligned} P(R) &= P(B_1 \cap R) + P(B_2 \cap R) + P(B_3 \cap R) \\ &= P(B_1)P(R \mid B_1) + P(B_2)P(R \mid B_2) + P(B_3)P(R \mid B_3) \\ &= \frac{1}{3}\cdot\frac{2}{6} + \frac{1}{6}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{5}{9} = \frac{4}{9}. \end{aligned}$$

Suppose now that the outcome of the experiment is a red chip, but we do not know from which bowl it was drawn. Accordingly, we compute the conditional probability that the chip was drawn from bowl B_1, namely, $P(B_1 \mid R)$. From the definition of conditional probability and the preceding result, we have

$$\begin{aligned} P(B_1 \mid R) &= \frac{P(B_1 \cap R)}{P(R)} \\ &= \frac{P(B_1)P(R \mid B_1)}{P(B_1)P(R \mid B_1) + P(B_2)P(R \mid B_2) + P(B_3)P(R \mid B_3)} \\ &= \frac{(1/3)(2/6)}{(1/3)(2/6) + (1/6)(1/3) + (1/2)(5/9)} = \frac{2}{8}. \end{aligned}$$

Similarly,

$$P(B_2 \mid R) = \frac{P(B_2 \cap R)}{P(R)} = \frac{(1/6)(1/3)}{4/9} = \frac{1}{8}$$

EXERCISES

1.6-1. Bowl B_1 contains two white chips, bowl B_2 contains two red chips, bowl B_3 contains two white and two red chips, and bowl B_4 contains three white chips and one red chip. The probabilities of selecting bowl B_1, B_2, B_3, or B_4 are 1/2, 1/4, 1/8, and 1/8, respectively. A bowl is selected using these probabilities and a chip is then drawn at random. Find

(a) $P(W)$, the probability of drawing a white chip.

(b) $P(B_1 \mid W)$, the conditional probability that bowl B_1 had been selected, given that a white chip was drawn.

1.6-2. Bean seeds from supplier A have an 85% germination rate and those from supplier B have a 75% germination rate. A seed-packaging company purchases 40% of its bean seeds from supplier A and 60% from supplier B and mixes these seeds together.

(a) Find the probability $P(G)$ that a seed selected at random from the mixed seeds will germinate.

(b) Given that a seed germinates, find the probability that the seed was purchased from supplier A.

1.6-3. A doctor is concerned about the relationship between blood pressure and irregular heartbeats. Among her patients, she classifies blood pressures as high, normal, or low and heartbeats as regular or irregular and finds that (a) 16% have high blood pressure; (b) 19% have low blood pressure; (c) 17% have an irregular heartbeat; (d) of those with an irregular heartbeat, 35% have high blood pressure; and (e) of those with normal blood pressure, 11% have an irregular heartbeat. What percentage of her patients have a regular heartbeat and low blood pressure?

1.6-4. Assume that an insurance company knows the following probabilities relating to automobile accidents:

Age of Driver	Probability of Accident	Fraction of Company's Insured Drivers
16–25	0.05	0.10
26–50	0.02	0.55
51–65	0.03	0.20
66–90	0.04	0.15

A randomly selected driver from the company's insured drivers has an accident. What is the conditional probability that the driver is in the 16–25 age group?

1.6-5. At a hospital's emergency room, patients are classified and 20% of them are critical, 30% are serious, and 50% are stable. Of the critical ones, 30% die; of the serious, 10% die; and of the stable, 1% die. Given that a patient dies, what is the conditional probability that the patient was classified as critical.

1.6-6. A life insurance company issues standard, preferred, and ultrapreferred policies. Of the company's policyholders of a certain age, 60% have standard policies and a probability of 0.01 of dying in the next year, 30% have preferred policies and a probability of 0.008 of dying in the next year, and 10% have ultrapreferred policies and a probability of 0.007 of dying in the next year. A policyholder of that age dies in the next year. What are the conditional probabilities of the deceased having had a standard, a preferred, and an ultrapreferred policy?

1.6-7. Among 60-year-old college professors, 90% are nonsmokers and 10% are smokers. The probability of a nonsmoker dying in the next year is 0.005 and the probability for smokers is 0.05. Given that one of this group of college professors dies in the next year, what is the conditional probability that the professor was a smoker?

1.6-8. A store sells four brands of DVD players. The least expensive brand, B_1, accounts for 40% of the sales. The other brands (in order of their price) have the following percentages of sales: B_2, 30%; B_3, 20%; and B_4, 10%. The respective probabilities of needing repair during warranty are 0.10 for B_1, 0.05 for B_2, 0.03 for B_3, and 0.02 for B_4. A randomly selected purchaser has a DVD player that needs repair under warranty. What are the four conditional probabilities of being brand B_i, $i = 1, 2, 3, 4$?

1.6-9. There is a new diagnostic test for a disease that occurs in about 0.05% of the population. The test is not perfect, but will detect a person with the disease 99% of the time. It will, however, say that a person without the disease has the disease about 3% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What are the conditional probabilities that

(a) the person has the disease?

(b) the person does not have the disease?

Discuss. HINT: Note that the fraction 0.0005 of diseased persons in the population is much smaller than the error probabilities of 0.01 and 0.03.

1.6-10. Suppose we want to investigate the percentage of abused children in a certain population. To do this, doctors examine some of these children taken at random from that population. However, doctors are not perfect: They sometimes classify an abused child (A) as one not abused (ND) or they classify a nonabused child (N) as one that is abused (AD). Suppose these error rates are $P(ND \mid A) = 0.08$ and $P(AD \mid N) = 0.05$, respectively; thus, $P(AD \mid A) = 0.92$ and $P(ND \mid N) = 0.95$ are the probabilities of the correct decisions. Let us pretend that only 2 percent of all children are abused; that is, $P(A) = 0.02$ and $P(N) = 0.98$.

(a) Select a child at random. What is the probability that the doctor classifies this child as abused. That is, compute

$$P(AD) = P(A)P(AD \mid A) + P(N)P(AD \mid N).$$

(b) Given that the child is classified by the doctor as abused, compute $P(N \mid AD)$ and $P(A \mid AD)$.

(c) Also, compute $P(N \mid ND)$ and $P(A \mid ND)$.

(d) Are the probabilities in (b) and (c) alarming? This happens because the error rates of 0.08 and 0.05 are high relative to the fraction 0.02 of abused children in the population.

1.6-11. At the beginning of a certain study of a group of persons, 15% were classified as heavy smokers, 30% as light smokers, and 55% as nonsmokers. In the five-year study, it was determined that the death rates of the heavy and light smokers were five and three times that of the nonsmokers, respectively. A randomly selected participant died over the five-year period; calculate the probability that the participant was a nonsmoker.

1.6-12. A test indicates the presence of a particular disease 90% of the time when the disease is present and the presence of the disease 2% of the time when the disease is not present. If 0.5% of the population has the disease, calculate the conditional probability that a person selected at random has the disease if the test indicates the presence of the disease.

1.6-13. A hospital receives two-fifths of its flu vaccine from Company A and the remainder from Company B. Each shipment contains a large number of vials of vaccine. From Company A, 3% of the vials are ineffective; from Company B, 2% are ineffective. A hospital tests $n = 25$ randomly selected vials from one shipment and finds that 2 are ineffective. What is the conditional probability that this shipment came from Company A?

1.6-14. Two processes of a company produce rolls of materials: The rolls of Process I are 3% defective and the rolls of Process II are 1% defective. Process I produces 60% of the company's output, Process II 40%. A roll is selected at random from the total output. Given that this roll is defective, what is the conditional probability that it is from Process I?

1.6-15. A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not about 5% of the time. The chemist produces compounds with the impurity about 20% of the time; that is, 80% do not have the impurity. A compound is selected at random from the chemist's output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has an impurity?

HISTORICAL COMMENTS Most probabilists would say that the mathematics of probability began when, in 1654, Chevalier de Méré, a French nobleman who liked to gamble, challenged Blaise Pascal to explain a puzzle and a problem created from his observations concerning rolls of dice. Of course, there was gambling well before this, and actually, almost 200 years before this challenge, a Franciscan monk, Luca Paccioli, proposed the same puzzle. Here it is:

> A and B are playing a fair game of balla. They agree to continue until one has six rounds. However, the game actually stops when A has won five and B three. How should the stakes be divided?

And over 100 years before de Méré's challenge, a 16th-century doctor, Girolamo Cardano, who was also a gambler, had figured out the answers to many dice problems,