EXERCISES

4.1-1. Let the joint p.m.f. of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1,2,\ y = 1,2,3,4.$$

(a) Find $f_1(x)$, the marginal p.m.f. of X.
(b) Find $f_2(y)$, the marginal p.m.f. of Y.
(c) Find $P(X > Y)$.
(d) Find $P(Y = 2X)$.
(e) Find $P(X + Y = 3)$.
(f) Find $P(X \le 3 - Y)$.
(g) Are X and Y independent or dependent? Why or why not?

4.1-2. Roll a pair of four-sided dice, one red and one black, each of which has possible outcomes 1, 2, 3, 4 that have equal probabilities. Let X equal the outcome on the red die, and let Y equal the outcome on the black die.

(a) On graph paper, show the space of X and Y.
(b) Define the joint p.m.f. on the space (similar to Figure 4.1-1).
(c) Give the marginal p.m.f. of X in the margin.
(d) Give the marginal p.m.f. of Y in the margin.
(e) Are X and Y dependent or independent? Why or why not?

4.1-3. Roll a pair of four-sided dice, one red and one black. Let X equal the outcome on the red die and let Y equal the sum of the two dice.

(a) On graph paper, describe the space of X and Y.
(b) Define the joint p.m.f. on the space (similar to Figure 4.1-1).
(c) Give the marginal p.m.f. of X in the margin.
(d) Give the marginal p.m.f. of Y in the margin.
(e) Are X and Y dependent or independent? Why or why not?

4.1-4. Select an (even) integer randomly from the set $\{0, 2, 4, 6, 8\}$. Then select an integer randomly from the set $\{0, 1, 2, 3, 4\}$. Let X equal the integer that is selected from the first set and let Y equal the sum of the two integers.

(a) Show the joint p.m.f. of X and Y on the space of X and Y.
(b) Compute the marginal p.m.f.'s.
(c) Are X and Y independent? Why or why not?

4.1-5. A particle starts at (0,0) and moves in one-unit independent steps with equal probabilities of 1/4 in each of the four directions: north, south, east, and west. Let S equal the east–west position and T the north–south position after n steps.

(a) Define the joint p.m.f. of S and T with $n = 3$. On a two–dimensional graph, give the probabilities of the joint p.m.f. and the marginal p.m.f.'s (similar to Figure 4.1-1).
(b) Let $X = S + 3$ and let $Y = T + 3$. How are X and Y distributed?

4.1-6. The torque required to remove bolts in a steel plate is rated as very high, high, average, and low, and these occur about 30%, 40%, 20%, and 10% of the time, respectively. Suppose $n = 25$ bolts are rated; what is the probability of rating 7 very high, 8 high, 6 average, and 4 low? Assume independence of the 25 trials.

Continuous **4.1-7.** Two construction companies make bids of X and Y (in \$100,000's) on a remodeling project. The joint p.d.f. of X and Y is uniform on the space $2 < x < 2.5, 2 < y < 2.3$. If X and Y are within 0.1 of each other, the companies will be asked to rebid; otherwise the low bidder will be awarded the contract. What is the probability that they will be asked to rebid?

Don't even need to integrate, just take ratio of areas (subtract two triangles).

4.1-8. In a smoking survey among boys between the ages of 12 and 17, 78% prefer to date nonsmokers, 1% prefer to date smokers, and 21% don't care. Suppose seven such boys are selected randomly. Let X equal the number who prefer to date nonsmokers and Y equal the number who prefer to date smokers.

(a) Determine the joint p.m.f. of X and Y. Be sure to include the support of the p.m.f.

(b) Find the marginal p.m.f. of X. Again include the support.

4.1-9. A manufactured item is classified as good, a "second," or defective with probabilities 6/10, 3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.

(a) Give the joint p.m.f. of X and Y, $f(x, y)$.

(b) Sketch the set of points for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?

(c) Find $P(X = 10, Y = 4)$.

(d) Give the marginal p.m.f. of X.

(e) Find $P(X \le 11)$.

4.1-10. Let $f(x, y) = 3/2, x^2 \le y \le 1, 0 \le x \le 1$, be the joint p.d.f. of X and Y.

(a) Find $P(0 \le X \le 1/2)$.

(b) Find $P(1/2 \le Y \le 1)$.

(c) Find $P(1/2 \le X \le 1, 1/2 \le Y \le 1)$.

(d) Find $P(X \ge 1/2, Y \ge 1/2)$.

(e) Are X and Y independent? Why or why not?

4.1-11. Continuous Let $f(x, y) = 2e^{-x-y}, 0 \le x \le y < \infty$, be the joint p.d.f. of X and Y. Find $f_1(x)$ and $f_2(y)$, the marginal p.d.f.'s of X and Y, respectively. Are X and Y independent?

4.1-12. Let X and Y have the joint p.d.f. $f(x, y) = x + y$, $0 \le x \le 1, 0 \le y \le 1$.

(a) Find the marginal p.d.f.'s $f_1(x)$ and $f_2(y)$ and show that $f(x, y) \not\equiv f_1(x)f_2(y)$. Thus, X and Y are dependent.

(b) Compute (i) μ_X; (ii) μ_Y; (iii) σ_X^2; and (iv) σ_Y^2.

4.1-13. Continuous Let $f(x, y) = (3/16)xy^2, 0 \le x \le 2, 0 \le y \le 2$, be the joint p.d.f. of X and Y. Find $f_1(x)$ and $f_2(y)$, the marginal probability density functions. Are the two random variables independent? Why or why not?

4.1-14. Let T_1 and T_2 be random times for a company to complete two steps in a certain process. Say T_1 and T_2 are measured in days and they have the joint p.d.f. that is uniform over the space $1 < t_1 < 10, 2 < t_2 < 6, t_1 + 2t_2 < 14$. What is $P(T_1 + T_2 > 10)$?

4.1-15. Continuous Let $f(x, y) = 4/3, \ 0 < x < 1, x^3 < y < 1$, zero elsewhere.

(a) Sketch the region where $f(x, y) > 0$.

(b) Find $P(X > Y)$.

EXERCISES

4.2-1. Let the random variables X and Y have the joint p.m.f

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1,2, \; y = 1,2,3,4.$$

Find the means μ_X and μ_Y, the variances σ_X^2 and σ_Y^2, and the correlation coefficient ρ. Are X and Y independent or dependent?

4.2-2. Let X and Y have the joint p.m.f. defined by $f(0,0) = f(1,2) = 0.2, f(0,1) = f(1,1) = 0.3$.

(a) Depict the points and corresponding probabilities on a graph.

(b) Give the marginal p.m.f.'s in the "margins."

(c) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, $\mathrm{Cov}(X, Y)$, and ρ.

(d) Find the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-3. Roll a fair four-sided die twice. Let X equal the outcome on the first roll and let Y equal the sum of the two rolls.

(a) Display the joint p.m.f. on a graph along with the marginal probabilities.

(b) Determine μ_X, μ_Y, σ_X^2, σ_Y^2, $\text{Cov}(X, Y)$, and ρ.

(c) Find the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-4. Show that, in the bivariate situation, E is a linear or distributive operator. That is, show that

$$E[a_1u_1(X_1, X_2) + a_2u_2(X_1, X_2)] \\ = a_1E[u_1(X_1, X_2)] + a_2E[u_2(X_1, X_2)].$$

4.2-5. Let X and Y be random variables with respective means μ_X and μ_Y, respective variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. Fit the line $y = a + bx$ by the method of least squares to the probability distribution by minimizing the expectation

$$K(a,b) = E[(Y - a - bX)^2]$$

with respect to a and b. HINT: Set $\partial K/\partial a = 0$ and $\partial K/\partial b = 0$, and solve simultaneously.

4.2-6. Let X and Y have a trinomial distribution with parameters $n = 3$, $p_1 = 1/6$, and $p_2 = 1/2$. Find

(a) $E(X)$.

(b) $E(Y)$.

(c) $\text{Var}(X)$.

(d) $\text{Var}(Y)$.

(e) $\text{Cov}(X, Y)$.

(f) ρ.

Note that $\rho = -\sqrt{p_1p_2/(1 - p_1)(1 - p_2)}$ in this case. (Indeed, the formula holds in general for the trinomial distribution; see Example 4.3-3.)

4.2-7. Let the joint p.m.f. of X and Y be

$$f(x,y) = 1/4, \\ (x,y) \in S = \{(0,0), (1,1), (1,-1), (2,0)\}.$$

(a) Are X and Y independent?

(b) Calculate $\text{Cov}(X, Y)$ and ρ.

This exercise also illustrates the fact that dependent random variables can have a correlation coefficient of zero.

4.2-8. The joint p.m.f. of X and Y is $f(x,y) = 1/6$, $0 \le x + y \le 2$, where x and y are nonnegative integers.

(a) Sketch the support of X and Y.

(b) Record the marginal p.m.f.'s $f_1(x)$ and $f_2(y)$ in the "margins."

(c) Compute $\text{Cov}(X, Y)$.

(d) Determine ρ, the correlation coefficient.

(e) Find the best-fitting line and draw it on your figure.

4.2-9. A certain raw material is classified as to moisture content X (in percent) and impurity Y (in percent). Let X and Y have the joint p.m.f. given by

	x			
y	1	2	3	4
2	0.10	0.20	0.30	0.05
1	0.05	0.05	0.15	0.10

(a) Find the marginal p.m.f.'s, the means, and the variances.

(b) Find the covariance and the correlation coefficient of X and Y.

(c) If additional heating is needed with high moisture content and additional filtering with high impurity such that the additional cost is given by the function $C = 2X + 10Y^2$ in dollars, find $E(C)$.

4.2-10. Let X and Y be random variables of the continuous type having the joint p.d.f.

$$f(x,y) = 2, \qquad 0 \le y \le x \le 1.$$

Draw a graph that illustrates the domain of this p.d.f.

(a) Find the marginal p.d.f.'s of X and Y.

(b) Compute μ_X, μ_Y, σ_X^2, σ_Y^2, $\text{Cov}(X, Y)$, and ρ.

(c) Determine the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-11. Let X and Y be random variables of the continuous type having the joint p.d.f.

$$f(x,y) = x + y, \qquad 0 < x < 1,\ 0 < y < 1.$$

Draw a graph that illustrates the domain of this p.d.f.

(a) Find the marginal p.d.f.'s of X and Y.

(b) Compute μ_X, μ_Y, σ_X^2, σ_Y^2, $\text{Cov}(X, Y)$, and ρ.

(c) Determine the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-12. Let X and Y be random variables of the continuous type having the joint p.d.f.

$$f(x,y) = 8xy, \qquad 0 \le x \le y \le 1.$$

Draw a graph that illustrates the domain of this p.d.f.

(a) Find the marginal p.d.f.'s of X and Y.
(b) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, $\text{Cov}(X, Y)$, and ρ.
(c) Determine the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-13. A car dealer sells X cars each day and always tries to sell an extended warranty on each of these cars. (In our opinion, most of these warranties are not good deals.) Let Y be the number of extended warranties sold; then $Y \le X$. The joint p.m.f. of X and Y is given by

$$f(x, y) = c(x + 1)(4 - x)(y + 1)(3 - y),$$

$$x = 0, 1, 2, 3; \ y = 0, 1, 2 \ \text{with} \ y \le x.$$

(a) Find the value of c.
(b) Sketch the support of X and Y.
(c) Record the marginal p.m.f.'s $f_1(x)$ and $f_2(y)$ in the "margins."
(d) Are X and Y independent?
(e) Compute μ_X and σ_X^2.
(f) Compute μ_Y and σ_Y^2.
(g) Compute $\text{Cov}(X, Y)$.
(h) Determine ρ, the correlation coefficient.
(i) Find the best-fitting line and draw it on your figure.

EXERCISES

4.3-1. Let X and Y have the joint p.m.f.

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \qquad y = 1, 2, 3, 4.$$

(a) Display the joint p.m.f. and the marginal p.m.f.'s on a graph like Figure 4.3-1(a).

(b) Find $g(x \mid y)$ and draw a figure like Figure 4.3-1(b), depicting the conditional p.m.f.'s for $y = 1, 2, 3$, and 4.

(c) Find $h(y \mid x)$ and draw a figure like Figure 4.3-1(c), depicting the conditional p.m.f.'s for $x = 1$ and 2.

(d) Find $P(1 \le Y \le 3 \mid X = 1)$, $P(Y \le 2 \mid X = 2)$, and $P(X = 2 \mid Y = 3)$.

(e) Find $E(Y \mid X = 1)$ and $\mathrm{Var}(Y \mid X = 1)$.

4.3-2. Let the joint p.m.f. $f(x, y)$ of X and Y be given by the following:

(x, y)	$f(x, y)$
(1, 1)	3/8
(2, 1)	1/8
(1, 2)	1/8
(2, 2)	3/8

Find the two conditional probability mass functions and the corresponding means and variances.

4.3-3. Let W equal the weight of laundry soap in a 1-kilogram box that is distributed in Southeast Asia. Suppose that $P(W < 1) = 0.02$ and $P(W > 1.072) = 0.08$. Call a box of soap light, good, or heavy, depending on whether $\{W < 1\}$, $\{1 \le W \le 1.072\}$, or $\{W > 1.072\}$, respectively. In $n = 50$ independent observations of these boxes, let X equal the number of light boxes and Y the number of good boxes.

(a) What is the joint p.m.f. of X and Y?

(b) Give the name of the distribution of Y along with the values of the parameters of this distribution.

(c) Given that $X = 3$, how is Y distributed conditionally?

(d) Determine $E(Y \mid X = 3)$.

(e) Find ρ, the correlation coefficient of X and Y.

4.3-4. The genes for eye color in a certain male fruit fly are (R, W). The genes for eye color in the mating female fruit fly are (R, W). Their offspring receive one gene for eye color from each parent. If an offspring ends up with either (R, R), (R, W), or

(W, R), its eyes will look red. Let X equal the number of offspring having red-eyes. Let Y equal the number of red eyed offspring having (R, W) or (W, R) genes.

(a) If the total number of offspring is $n = 400$, how is X distributed?
(b) Give the values of $E(X)$ and $\text{Var}(X)$.
(c) Given that $X = 300$, how is Y distributed?
(d) Give the values of $E(Y \mid X = 300)$ and $\text{Var}(Y \mid X = 300)$.

4.3-5. Let X and Y have a trinomial distribution with $n = 2$, $p_1 = 1/4$, and $p_2 = 1/2$.

(a) Give $E(Y \mid x)$.
(b) Compare your answer in part (a) with the equation of the line of best fit in Example 4.2-2. Are they the same? Why or why not?

4.3-6. An insurance company sells both homeowners insurance and automobile deductible insurance. Let X be the deductible on the homeowners insurance and Y the deductible on automobile insurance. Among those who take both types of insurance with this company, we find the following probabilities:

	x		
y	100	500	1000
1000	0.05	0.10	0.15
500	0.10	0.20	0.05
100	0.20	0.10	0.05

(a) Compute the following probabilities: $P(X = 500)$, $P(Y = 500)$, $P(Y = 500 \mid X = 500)$, $P(Y = 100 \mid X = 500)$.
(b) Compute the means μ_X, μ_Y, and the variances σ_X^2, σ_Y^2.
(c) Compute the conditional means $E(X \mid Y = 100)$, $E(Y \mid X = 500)$.
(d) Compute $\text{Cov}(X, Y)$.
(e) Compute the correlation coefficient $\rho = \text{Cov}(X, Y)/\sigma_X\sigma_Y$.

4.3-7. Using the joint p.m.f. from Exercise 4.2-3, find the value of $E(Y \mid x)$ for $x = 1, 2, 3, 4$. Do the points $[x, E(Y \mid x)]$ lie on the best-fitting line?

4.3-8. An unbiased six-sided die is cast 30 independent times. Let X be the number of ones and Y the number of twos.

(a) What is the joint p.m.f. of X and Y?
(b) Find the conditional p.m.f. of X, given $Y = y$.
(c) Compute $E(X^2 - 4XY + 3Y^2)$.

4.3-9. Let X and Y have a uniform distribution on the set of points with integer coordinates in $S = \{(x, y) : 0 \le x \le 7, x \le y \le x + 2\}$. That is, $f(x, y) = 1/24$, $(x, y) \in S$, and both x and y are integers. Find

(a) $f_1(x)$.
(b) $h(y \mid x)$.
(c) $E(Y \mid x)$.
(d) $\sigma^2_{Y|x}$.
(e) $f_2(y)$.

4.3-10. Let $f_1(x) = 1/10$, $x = 0, 1, 2, \ldots, 9$, and $h(y \mid x) = 1/(10 - x)$, $y = x, x + 1, \ldots, 9$. Find

(a) $f(x, y)$.
(b) $f_2(y)$.
(c) $E(Y \mid x)$.

4.3-11. An automobile repair shop makes an initial estimate X (in thousands of dollars) of the amount of money needed to fix a car after an accident. Say X has the p.d.f.

$$f(x) = 2e^{-2(x-0.2)}, \qquad 0.2 < x < \infty.$$

Given that $X = x$, the final payment Y has a uniform distribution between $x - 0.1$ and $x + 0.1$. What is the expected value of Y?

4.3-12. For the random variables defined in Example 4.3-5, calculate the correlation coefficient directly from the definition

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X\sigma_Y}.$$

4.3-13. Let $f(x, y) = 1/40$, $0 \le x \le 10$, $10 - x \le y \le 14 - x$, be the joint p.d.f. of X and Y.

(a) Sketch the region for which $f(x, y) > 0$.
(b) Find $f_1(x)$, the marginal p.d.f. of X.
(c) Determine $h(y \mid x)$, the conditional p.d.f. of Y, given that $X = x$.
(d) Calculate $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

4.3-14. Let $f(x, y) = 1/8$, $0 \le y \le 4$, $y \le x \le y + 2$, be the joint p.d.f. of X and Y.

(a) Sketch the region for which $f(x, y) > 0$.
(b) Find $f_1(x)$, the marginal p.d.f. of X.
(c) Find $f_2(y)$, the marginal p.d.f. of Y.
(d) Determine $h(y \mid x)$, the conditional p.d.f. of Y, given that $X = x$.
(e) Determine $g(x \mid y)$, the conditional p.d.f. of X, given that $Y = y$.
(f) Compute $y = E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

(g) Compute $x = E(X \mid y)$, the conditional mean of X, given that $Y = y$.

(h) Graph $y = E(Y \mid x)$ on your sketch in part (a). Is $y = E(Y \mid x)$ linear?

(i) Graph $x = E(X \mid y)$ on your sketch in part (a). Is $x = E(X \mid y)$ linear?

4.3-15. Let X have a uniform distribution $U(0,2)$, and let the conditional distribution of Y, given that $X = x$, be $U(0,x^2)$.

(a) Determine $f(x,y)$, the joint p.d.f. of X and Y.

(b) Calculate $f_2(y)$, the marginal p.d.f. of Y.

(c) Compute $E(X \mid y)$, the conditional mean of X, given that $Y = y$.

(d) Find $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

4.3-16. Let X have a uniform distribution on the interval $(0, 1)$. Given that $X = x$, let Y have a uniform distribution on the interval $(0, x)$.

(a) Define the conditional p.d.f. of Y, given that $X = x$. Be sure to include the domain.

(b) Find $E(Y \mid x)$.

(c) Determine the joint p.d.f. of X and Y.

(d) Find the marginal p.d.f. of Y.

4.3-17. The marginal distribution of X is $U(0,1)$. The conditional distribution of Y, given that $X = x$, is $U(0, e^x)$.

(a) Determine $h(y \mid x)$, the conditional p.d.f. of Y, given that $X = x$.

(b) Find $E(Y \mid x)$.

(c) Display the joint p.d.f. of X and Y. Sketch the region where $f(x, y) > 0$.

(d) Find $f_2(y)$, the marginal p.d.f. of Y.

4.3-18. Let X have a uniform distribution on the interval $(0,1)$. Given that $X = x$, let Y have a uniform distribution on the interval $(0, x + 1)$.

(a) Find the joint p.d.f. of X and Y. Sketch the region where $f(x,y) > 0$.

(b) Find $E(Y \mid x)$, the conditional mean of Y, given that $X = x$. Draw this line on the region sketched in part (a).

(c) Find $f_2(y)$, the marginal p.d.f. of Y. Be sure to include the domain.

4.3-19. Let X and Y have the joint p.d.f. $f(x, y) = cx(1 - y)$, $0 < y < 1$ and $0 < x < 1 - y$.

(a) Determine c.

(b) Compute $P(Y < X \mid X \le 1/4)$.

4.3-20. Select x and y to create a triangle of perimeter 1 that has sides of lengths x, y, and $1 - x - y$. By Heron's formula, the area of such a triangle is

$$T = \frac{1}{4}\sqrt{(2x + 2y - 1)(1 - 2x)(1 - 2y)}.$$

If x and y are values of the jointly distributed random variables X and Y, then T is a random variable that can be thought of as the area of a "random triangle."

(a) Determine the possible values of (X, Y) and graph this region in the xy-plane. Call the region R.

(b) Select the random point (X, Y) uniformly from R. That is, the joint p.d.f. of (X, Y) is $f(x, y) = 1/A(R)$, where $A(R)$ is the area of R. Show that $E(T) = \pi/105$ and the variance of T is

$$E(T^2) - [E(T)]^2 = 1/960 - (\pi/105)^2.$$

HINT: Use Maple or some other computer algebra system.

(c) Find the marginal p.d.f. of X and the conditional p.d.f. of Y, given that $X = x$.

(d) Simulate 5000 pairs of observations of X and Y and then calculate the areas of the simulated triangles. Show that the sample mean and the sample variance of the areas of these 5000 triangles are close to the theoretical values.

(e) Now select (X, Y) as follows: The random variable X is selected randomly. That is, $f_1(x) = c$, where c is selected appropriately. Be sure to define the domain for this p.d.f. Given that $X = x$, Y is selected randomly (uniformly) from the appropriate interval. Define the joint p.d.f. of X and Y [the domain is R from part (a)], and again find $E(T)$ and the variance of T. Compare the theoretical and the sample values.

Remark This exercise is based on research at Hope College by students Andrea Douglass, Courtney Fitzgerald, and Scott Mihalik. (See references.) ■

Thus, in the bivariate normal case, $\rho = 0$ does imply independence of X and Y. Note that these characteristics of the bivariate normal distribution can be extended to the trivariate normal distribution or, more generally, the multivariate normal distribution. This is done in more advanced texts that assume some knowledge of matrices [e.g., Hogg, McKean, and Craig (2005)].

EXERCISES

4.4-1. Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3$, $\mu_Y = 10$, $\sigma_X^2 = 25$, $\sigma_Y^2 = 9$, and $\rho = 3/5$. Compute

(a) $P(-5 < X < 5)$.

skip→(b) $P(-5 < X < 5 \mid Y = 13)$.

(c) $P(7 < Y < 16)$.

(d) $P(7 < Y < 16 \mid X = 2)$.

4.4-2. Show that the expression in the exponent of Equation 4.4-2 is equal to the function $q(x, y)$ given in the text.

4.4-3. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 2.8$, $\mu_Y = 110$, $\sigma_X^2 = 0.16$, $\sigma_Y^2 = 100$, and $\rho = 0.6$. Compute

(a) $P(106 < Y < 124)$.

(b) $P(106 < Y < 124 \mid X = 3.2)$.

4.4-4. Let X and Y have a bivariate normal distribution with $\mu_X = 70$, $\sigma_X^2 = 100$, $\mu_Y = 80$, $\sigma_Y^2 = 169$, and $\rho = 5/13$. Find

(a) $E(Y \mid X = 72)$.

(b) $\text{Var}(Y \mid X = 72)$.

(c) $P(Y \le 84 \mid X = 72)$.

4.4-5. Let X denote the height in centimeters and Y the weight in kilograms of male college students. Assume that X and Y have a bivariate normal distribution with parameters $\mu_X = 185$, $\sigma_X^2 = 100$, $\mu_Y = 84$, $\sigma_Y^2 = 64$, and $\rho = 3/5$.

(a) Determine the conditional distribution of Y, given that $X = 190$.

(b) Find $P(86.4 < Y < 95.36 \mid X = 190)$.

4.4-6. For a freshman taking introductory statistics and majoring in psychology, let X equal the student's ACT mathematics score and Y the student's ACT verbal score. Assume that X and Y have a bivariate normal distribution with $\mu_X = 22.7$, $\sigma_X^2 = 17.64$, $\mu_Y = 22.7$, $\sigma_Y^2 = 12.25$, and $\rho = 0.78$.

(a) Find $P(18.5 < Y < 25.5)$.

(b) Find $E(Y \mid x)$.

(c) Find $\text{Var}(Y \mid x)$.

(d) Find $P(18.5 < Y < 25.5 \mid X = 23)$.

(e) Find $P(18.5 < Y < 25.5 \mid X = 25)$.

(f) For $x = 21, 23$, and 25, draw a graph of $z = h(y \mid x)$ similar to Figure 4.4-1.

4.4-7. For a pair of gallinules, let X equal the weight in grams of the male and Y the weight in grams of the female. Assume that X and Y have a bivariate normal distribution with $\mu_X = 415$, $\sigma_X^2 = 611$, $\mu_Y = 347$, $\sigma_Y^2 = 689$, and $\rho = -0.25$. Find

(a) $P(309.2 < Y < 380.6)$.

(b) $E(Y \mid x)$.

(c) $\text{Var}(Y \mid x)$.

(d) $P(309.2 < Y < 380.6 \mid X = 385.1)$.

4.4-8. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 10$, $\sigma_X^2 = 9$, $\mu_Y = 15$, $\sigma_Y^2 = 16$, and $\rho = 0$. Find

(a) $P(13.6 < Y < 17.2)$.

(b) $E(Y \mid x)$.

(c) $\text{Var}(Y \mid x)$.

(d) $P(13.6 < Y < 17.2 \mid X = 9.1)$.

4.4-9. Let X and Y have a bivariate normal distribution. Find two different lines, $a(x)$ and $b(x)$, parallel to and equidistant from $E(Y \mid x)$, such that

$$P[a(x) < Y < b(x) \mid X = x] = 0.9544$$

for all real x. Plot $a(x)$, $b(x)$, and $E(Y \mid x)$ when $\mu_X = 2$, $\mu_Y = -1$, $\sigma_X = 3$, $\sigma_Y = 5$, and $\rho = 3/5$.

4.4-10. In a college health fitness program, let X denote the weight in kilograms of a male freshman at the beginning of the program and Y denote his weight change during a semester. Assume that X and Y have a bivariate normal distribution with $\mu_X = 72.30$, $\sigma_X^2 = 110.25$, $\mu_Y = 2.80$, $\sigma_Y^2 = 2.89$, and $\rho = -0.57$. (The lighter students tend to gain weight, while the heavier students tend to lose weight.) Find

(a) $P(2.80 \le Y \le 5.35)$.

(b) $P(2.76 \le y \le 5.34 \mid X = 82.3)$.

4.4-11. For a female freshman in a health fitness program, let X equal her percentage of body fat at the beginning of the program and Y equal the change in her percentage of body fat measured at the end of the program. Assume that X and Y have a bivariate normal distribution with $\mu_X = 24.5$, $\sigma_X^2 = 4.8^2 = 23.04$, $\mu_Y = -0.2$, $\sigma_Y^2 = 3.0^2 = 9.0$, and $\rho = -0.32$. Find

(a) $P(1.3 \le Y \le 5.8)$.
(b) $\mu_{Y|x}$, the conditional mean of Y, given that $X = x$.
(c) $\sigma^2_{Y|x}$, the conditional variance of Y, given that $X = x$.
(d) $P(1.3 \le Y \le 5.8 \mid X = 18)$.

4.4-12. For a male freshman in a health fitness program, let X equal his percentage of body fat at the beginning of the program and Y equal the change in his percentage of body fat measured at the end of the program. Assume that X and Y have a bivariate normal distribution with $\mu_X = 15.00$, $\sigma_X^2 = 4.5^2$, $\mu_Y = -1.55$, $\sigma_Y^2 = 1.5^2$, and $\rho = -0.60$. Find

(a) $P(0.205 \le Y \le 0.805)$.
(b) $P(0.21 \le Y \le 0.81 \mid X = 20)$.

4.4-13. The concentration (X) and the viscosity (Y) of a chemical product have a bivariate normal distribution with parameters $\mu_X = 3, \mu_Y = 2$, $\sigma_X = 2, \sigma_Y = 1$, and $\rho = 0.6$.

(a) What is $P(X + Y \ge 4)$? HINT: Find the moment-generating function of $Z = X + Y$ and show that it has a normal distribution.
(b) Compute the conditional probability $P(X \ge 3.5 \mid Y = 2.5)$.
(c) Compute

$$P\left\{\left(\frac{1}{0.64}\right)\left[\left(\frac{X-3}{2}\right)^2 - (2)(0.6)\left(\frac{X-3}{2}\right)(Y-2) + (Y-2)^2\right] \ge 5.99\right\}.$$

HINT: Show that the moment-generating function of the second degree function $q(X, Y)$ in the probability statement is equal to $(1 - 2t)^{-1}$.

4.4-14. Suppose that in a certain population of cigarettes the tar (X) per cigarette in milligrams and the nicotine (Y) have a bivariate normal distribution with parameters $\mu_X = 14.1, \sigma_X = 2.5, \mu_Y = 1.3, \sigma_Y = 0.1$, and $\rho = 0.8$. Compute

(a) $P(Y > 1.4 \mid X = 15)$,
(b) $P(X > 15 \mid Y = 1.4)$.

4.4-15. An obstetrician does ultrasound examinations on her patients between their 16th and 25th weeks of pregnancy to check the growth of each fetus. Let X equal the widest diameter of the fetal head, and let Y equal the length of the femur, both measurements in mm. Assume that X and Y have a bivariate normal distribution with $\mu_X = 60.6$, $\sigma_X = 11.2$, $\mu_Y = 46.8$, $\sigma_Y = 8.4$, and $\rho = 0.94$.

(a) Find $P(40.5 < Y < 48.9)$.
(b) Find $P(40.5 < Y < 48.9 \mid X = 68.6)$.