

Announcements

1) Reading for Thursday:
Section 3.2

2) HW 2 due Thursday. For

5), you may assume: if $m, n \in \mathbb{Z}$,

$m+n$ and $m \cdot n \in \mathbb{Z}$

Example 1: (orderings of n -element set)

Let $X = \{x_1, x_2, \dots, x_n\}$ be
an n -element set.

(Try small values of n : $n=3$

Picture: 3 spots

3 choices < 2 choice < 1 choice
1 2 3

Total number: $6 = 3 \cdot 2 \cdot 1 = 3!$)

In general: for n elements,
you have

n choices for the smallest

$(n-1)$ choices for the next largest

$(n-2)$ choices for the next largest

⋮

2 choices for 2nd largest

1 choice for largest

Total number of different orderings

= product of your choices

= $n!$



Equivalence Relations

(Section 7.2)

Definition: (direct product of sets)

If S, T are sets, the **direct product** $S \times T$ is the set of all ordered pairs (s, t) with $s \in S, t \in T$.

ordered: if it makes sense,

$$(s, t) \neq (t, s)$$

Example 2: $\{1, 2, 3\} \times \{3, 8, 9\}$

List the elements:

$\{ (1, 3), (1, 8), (1, 9),$
 $(2, 3), (2, 8), (2, 9)$
 $(3, 3), (3, 8), (3, 9) \}$

Definition : (equivalence relation)

An equivalence relation on a set Y is a subset R of

$Y \times Y$ such that $\forall x, y, z \in Y$,

1) $(x, x) \in R$ (reflexivity)

2) IF $(x, y) \in R$, then $(y, x) \in R$
(symmetry)

3) IF $(x, y) \in R$ and $(y, z) \in R$, then
 $(x, z) \in R$ (transitivity)

Shorthand: Use " \sim " to indicate the equivalence. So $\forall x, y, z \in Y$

Reflexivity $x \sim x$

Symmetry if $x \sim y$, then $y \sim x$

Transitivity if $x \sim y$ and $y \sim z$,
then $x \sim z$

Example 3: (fractions) If

$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, we define

$\frac{a}{b} \sim \frac{c}{d}$ if $ad = bc$
(or $ad - bc = 0$)

Check:

Reflexivity is $\frac{a}{b} \sim \frac{a}{b}$?

$$ab = ab \quad \checkmark$$

Symmetry

Suppose

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{Then}$$

$$ad = bc, \text{ so by}$$

commutativity of multiplication,

$$\frac{c}{d} \sim \frac{a}{b} \quad \text{since}$$

$$cb = bc \text{ and } da = ad.$$

Transitivity Suppose $\frac{m}{n} \in \mathbb{Q}$ and

$$\frac{a}{b} \sim \frac{c}{d}, \quad \frac{c}{d} \sim \frac{m}{n}.$$

Then $ad = bc$ and

$$cn = dm.$$

$n, d, b \neq 0$, so (for example)

$$c = \frac{dm}{n}. \quad \text{Substituting}$$

into $ad = bc$, we get

$$ad = b \left(\frac{dm}{n} \right)$$

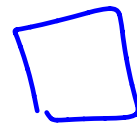
multiply both sides by n

$$a d n = b d m$$

Divide both sides by d

$$a n = b m$$

$$\Rightarrow \frac{a}{b} \sim \frac{m}{n}$$



Notation : (equivalence classes)

If Y is a set, $x \in Y$

and " \sim " is an equivalence relation on Y , the

equivalence class of x ,

denoted $[x]$ is

$$[x] = \{ z \in Y \mid x \sim z \}$$

Example 4: On $M_2(\mathbb{R})$ (2×2 matrices with real entries), define

$$A \sim B \quad \text{if} \quad \boxed{\det(AB) \geq 0}$$

Note: $\det(AB) = \det(A)\det(B)$.

Reflexivity: $\det(A \cdot A)$
 $= \det(A) \cdot \det(A)$

$$= (\det(A))^2 \geq 0 \quad \checkmark$$

Symmetry $B \sim A$ means

$$\det(BA) \geq 0. \quad \text{But}$$

$$\begin{aligned} \det(BA) &= \det(B)\det(A) \\ &= \det(A)\det(B) \\ &= \det(AB) \geq 0 \end{aligned}$$

if $A \sim B$, so $B \sim A$. ✓

Transitivity fails!

Translate: $A \sim B, B \sim C \Rightarrow A \sim C$

means

if $\det(AB) \geq 0$ and
 $\det(BC) \geq 0$, then
 $\det(AC) \geq 0$.

Choose A, B, C such that
 $\det(AB), \det(BC) \geq 0$
but $\det(AC) < 0$.

Choose $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\det(AB) = \det(A)\det(B) = 0$$

$$\det(BC) = \det(B)\det(C) = 0$$

$$\det(AC) = \det(A)\det(C) = -1 < 0$$

Not an equivalence relation!