Announcements

1) Reading for Thursday:
   Section 3.2

2) HW 2 due Thursday. For
   5), you may assume: if $m, n \in \mathbb{Z}$,
   $m+n$ and $m \cdot n \in \mathbb{Z}$

Example 1: (orderings of n-element set)

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an n-element set.

(Try small values of n: $n=3$

Picture: 3 spots

$$\underset{1}{\underline{3 \text{ choices}}} \quad \underset{2}{\underline{2 \text{ choice}}} \quad \underset{3}{\underline{1 \text{ choice}}}$$

Total number: $6 = 3 \cdot 2 \cdot 1 = 3!$)

In general: for n elements,

you have

n choices for the smallest

$(n-1)$ choices for the next largest

$(n-2)$ choices for the next largest

$\vdots$

2 choices for 2^{nd} largest

1 choices for largest

Total number of different orderings

= product of your choices

$= n!$ $\square$

Equivalence Relations

(Section 7.2)

Definition: (direct product of sets)

If S, T are sets, the **direct product** $S \times T$ is the set of all ordered pairs (s,t) with $s \in S$, $t \in T$.

ordered: if it makes sense,

$$(s,t) \neq (t,s)$$

Example 2: $\{1,2,3\} \times \{3,8,9\}$

List the elements:

$$\{(1,3),(1,8),(1,9),$$
$$(2,3),(2,8),(2,9)$$
$$(3,3),(3,8),(3,9)\}$$

Definition: (equivalence relation)

An equivalence relation on a set Y is a subset R of $Y \times Y$ such that $\forall\, x, y, z \in Y$,

1) $(x, x) \in R$ (reflexivity)

2) If $(x, y) \in R$, then $(y, x) \in R$ (symmetry)

3) If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ (transitivity)

Shorthand: Use "$\sim$" to indicate the equivalence. So $\forall\, x, y, z \in Y$

Reflexivity $x \sim x$

Symmetry if $x \sim y$, then $y \sim x$

Transitivity if $x \sim y$ and $y \sim z$, then $x \sim z$

Example 3: (fractions) If

$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, we define

$$\frac{a}{b} \sim \frac{c}{d} \text{ if } \boxed{ad = bc}$$

(or $ad - bc = 0$)

Check:

Reflexivity is $\frac{a}{b} \sim \frac{a}{b}$?

$$ab = ab \checkmark$$

Symmetry Suppose

$$\frac{a}{b} \sim \frac{c}{d}.$$ Then

$ad = bc$, so by

commutativity of multiplication,

$$\frac{c}{d} \sim \frac{a}{b}$$ since

$cb = bc$ and $da = ad$.

Transitivity Suppose $\frac{m}{n} \in \mathbb{Q}$ and

$$\frac{a}{b} \sim \frac{c}{d}, \quad \frac{c}{d} \sim \frac{m}{n}.$$

Then $ad = bc$ and

$$cn = dm.$$

$n, d, b \neq 0$, so (for example)

$c = \dfrac{dm}{n}$. Substituting

into $ad = bc$, we get

$$ad = b\left(\frac{dm}{n}\right)$$

multiply both sides by n

$$adn = bdm$$

Divide both sides by d

$$an = bm$$

$$\Rightarrow \quad \frac{a}{b} \sim \frac{m}{n}$$

$\square$

Notation: (equivalence classes)

If Y is a set, $x \in Y$ and "$\sim$" is an equivalence relation on Y, the equivalence class of x, denoted $[x]$ is

$$\boxed{[x] = \{ z \in Y \mid x \sim z \}}$$

Example 4: On $M_2(\mathbb{R})$ (2×2 matrices with real entries), define

$$A \sim B \text{ if } \boxed{\det(AB) \geq 0}$$

Note: $\det(AB) = \det(A)\det(B)$.

Reflexivity: $\det(A \cdot A)$

$= \det(A) \cdot \det(A)$

$= (\det(A))^2 \geq 0$ ✓

Symmetry $B \sim A$ means

$\det(BA) \geq 0$. But

$$\det(BA) = \det(B)\det(A)$$
$$= \det(A)\det(B)$$
$$= \det(AB) \geq 0$$

if $A \sim B$, so $B \sim A$.

Transitivity Fails!

Translate: $A \sim B,\ B \sim C \Rightarrow A \sim C$

means

if $\det(AB) \geq 0$ and

$\det(BC) \geq 0$, then

$\det(AC) \geq 0$.

Choose A, B, C such that

$\det(AB), \det(BC) \geq 0$.

but $\det(AC) < 0$.

Choose $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\det(AB) = \det(A)\det(B) = 0$

$\det(BC) = \det(B)\det(C) = 0$

$\det(AC) = \det(A)\det(C) = -1 < 0$

Not an equivalence relation!