1.2 Properties of Probability

<u>Motivational Example</u> Consider the value of Facebook stock at the close of the NASDAQ tomorrow as compared to the value at the close of the NASDAQ today. I'm interested in whether it goes up, remains the same, or goes down. Thus, the set of outcomes of this "experiment" is $\{up, rem, down\}$. My <u>belief</u> is the following probability table:

outcome	probability
goes up	.1
remains same	.2
goes down	.7

$P(\{up\}) =$ probability it goes up $= .1$

$P(\{up, rem\}) =$ probability of the event that it goes up <u>or</u> remains the same

$= P(\{up\}) + P(\{rem\}) = .1 + .2 = .3$

$P(\{rem, down\}) =$ prob. of the event that it remains the same or goes down

$= P(\{rem\}) + P(\{down\}) = .2 + .7 = .9$

Notice $P(\overline{\{up\}}) = .9 = 1 - P(\{up\})$

$P(\{up, rem, down\})$ = prob. that it goes up, remains the same, or goes down

$= P(\{up\}) + P(\{rem\}) + P(\{down\})$

$= .1 + .2 + .7$

$= 1$

Rem This example illustrates the key notions of what we will learn in this section: experiment
sample space
probability measure.

In this example: experiment = check the stock at close of NASDAQ tomorrow

sample space = $\{up, rem, down\}$

probability measure = this function $P(\ )$ which assigns numbers to subsets of the sample space in an "additive" way.

We first turn to experiments and sample spaces. The term experiment is used much more broadly in probability theory than in natural science.

Def An <u>experiment</u> is a process by which an observation (measurement) is made.

<u>Examples</u> of experiments are:

- controlled experiments in a lab, in the conventional sense of natural science
- computer simulation
- opinion poll
- looking at values in hospital records years later (retrospective experiment)
- clinical trials
- checking FB stock at tomorrow's close of the NASDAQ.
- the roll of a die
- the flip of a coin
- selecting a chip from a bowl
- randomly selecting an object from a specified collection.

Def The sample space S of an experiment is the set of all possible outcomes of the experiment. The sample space is also called the outcome space.

Ex 1) In the Facebook example above,

experiment = check FB stock at tomorrow's NASDAQ close

sample space $= S = \{up, rem, down\}$

2) experiment = measure your blood glucose level

sample space $= S =$ interval $[50 \underbrace{mg/dL}_{\text{milligrams/decilites}}, 140\ mg/dL]$

3) experiment = roll a die

sample space $= S = \{1,2,3,4,5,6\}$

Def An event in a sample space S is a subset of S.

Ex 1) $\{up\}$ and $\{up, rem\}$ are events

2) your level is 100 mg/dL

" " " greater than 100 mg/dL

" " " between 79.2 and 110 mg/dL. (normal)

3) $\{1\}$ is an event in $S=\{1,2,3,4,5,6\}$

$\{\text{the roll is odd}\} = \{1,3,5\}$ is an event

$\{\text{the roll is even}\} = \{2,4,6\}$ is an event.

<u>Def</u> The elements of the sample space are called <u>sample points</u>, or <u>outcomes</u>.

<u>Ex</u> 1) In $S = \{\text{up, rem, down}\}$ the sample points are up, rem, down.

2) In $S = [50\ mg/dL, 140\ mg/dL]$, an example of a sample point is $90\ mg/dL$.

3) In $S = \{1,2,3,4,5,6\}$, 4 is a sample point.

Considering that an event is a subset of a sample space, we may now re-interpret the set relations "$\subseteq$" and "$\in$" in terms of events occurring or not occurring.

Re-Interpretation of Set Relations "$\subseteq$" and "$\in$"

Let S be a sample space and $A, B \subseteq S$, $x \in S$, i.e. A and B are <u>events</u>, and x is an outcome.

$x \in A$ means "A occurs when x does"

$x \notin A$ means "A does not occur when x does"

$A \subseteq B$ means "if A occurs then so does B"

$A \cap B$ is "the event that both A and B occur"

$A \cup B$ is "the event that A or B or both occurs"

$\bar{A}$ is "the event that A does not occur"

$A \cap B = \emptyset$ means "A and B are <u>mutually exclusive</u>"
i.e. "A and B cannot both occur simultaneously"

Probability Measures

Recall We discussed an example about the price of Facebook stock. There were 3 possible outcomes for the experiment of checking the stock price: it goes up, remains the same, or goes down. My belief about the respective probabilities was .1, .2, .7.
From this, we computed the probabilities of events (event = subset of sample space) such as {up, rem} by

$$P(\{up, rem\}) = P(\{up\}) + P(\{rem\}) = .1 + .2 = .3.$$

Remark Notice that we had constructed a function $P(-)$ which assigned to each subset of the sample space a number, and this assignment was additive with respect to disjoint unions.
Here is the whole function:

event	P(event)
{up}	.1
{rem}	.2
{down}	.7
{up, rem}	.3
{up, down}	.8
{rem, down}	.9
{up, rem, down}	1
$\emptyset$	0

We now abstract the key properties for such a function with inputs as subsets of a sample space, and outputs real numbers in the interval $[0,1]$.

Def Let S be a sample space for an experiment. A probability measure P is a function

$$P : (\text{set of subsets of } S) \longrightarrow \text{real numbers}$$

such that

1) For each $A \subseteq S$, $P(A) \geq 0$
2) $P(S) = 1$
3) The axiom of countable additivity holds: For any pairwise mutually exclusive events $A_1, A_2, A_3, \ldots$ we have

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

Rem $A_1 \cup A_2 \cup A_3 \cup \cdots$ is also denoted $\bigcup_{i=1}^{\infty} A_i$

Rem The axiom of countable additivity implies the axiom of finite additivity, which says:

For any finite list $A_1, A_2, A_3, \ldots, A_k$ of pairwise mutually exclusive events we have

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k).$$

Ex If S is a sample space and we have an assignment of probabilities in $[0,1]$ to each sample point and these all add up to 1, then we have a probability measure on S in the following way. Let $A = \{a_1, a_2, \ldots, a_n\} \subseteq S$.

$$P(A) := P(\{a_1\}) + P(\{a_2\}) + \cdots + P(\{a_n\})$$

(and similary define the probability of an infinite subset as an infinite sum).

This was exactly how we did the Facebook example!

Ex For instance, if S is finite and each outcome is equally likely, then each individual outcome has probability $\frac{1}{|S|}$ and

$$P(A) := \frac{|A|}{|S|}$$

is a probability measure.

Let's see this in two examples: the roll of a die and three coin flips.

Ex Consider the experiment of rolling a balanced die.

Then sample space $= S = \{1,2,3,4,5,6\}$.

What is the probability of rolling <5?

$$P(\{1,2,3,4\}) = \frac{|\{1,2,3,4\}|}{|S|} = \frac{4}{6} = \frac{2}{3}$$

(The die is balanced, so each outcome is equally likely, so we can use the previous example).

Q: Recompute as a sum, compare two previous examples.

Ex Consider the experiment of doing a sequence of 3 coin flips.

Then sample space $= S = \{$ HHH, HHT,
HTH, HTT,
TTH, TTT,
THH, THT $\}$

A = event of at least two heads = $\{$HHH, HHT, HTH, THH$\}$

$$P(A) = \frac{|A|}{|S|} = \frac{4}{8} = \frac{1}{2}$$

B = event of a head on first and third toss = $\{$HHH, HTH$\}$

$$P(B) = \frac{|B|}{|S|} = \frac{2}{8} = \frac{1}{4}$$

What is the probability of exactly two heads?

The definition of probability measure has several logical consequences that are useful:

<u>Thm</u> Let P be a probability measure on a sample space S, and let A, B be events. Then

1) $P(\bar{A}) = 1 - P(A)$

2) $P(\emptyset) = 0$

3) $P(A) \leq 1$.

4) If $A \subseteq B$, then $P(A) \leq P(B)$.

<u>Ex</u> Consider these in the example of sequences of 3 coin flips above.

$A :=$ event of exactly 2 heads $= \{HHT, HTH, THH\}$

$\bar{A} :=$ event of 0, 1, or 3 heads $= \{TTT, HTT, THT, TTH, HHH\}$

$B :=$ event of at least 2 heads $= \{HHT, HTH, THH, HHH\}$

1) $P(\bar{A}) = 1 - P(A)$

$\frac{5}{8} = 1 - \frac{3}{8}$ ✓

2) $P(\emptyset) = \frac{|\emptyset|}{|S|} = \frac{0}{8} = 0$ ✓

3) $P(A) = \frac{3}{8} \leq 1$

4) $A \subseteq B$ above $P(A) \leq P(B)$ is $\frac{3}{8} \leq \frac{4}{8}$ ✓

This is what the Thm above says in this concrete example.

<u>Thm</u> (Additive Law for Probability, Inclusion-Exclusion Principle) Let P be a probability measure on a sample space S. Suppose $A, B \subseteq S$. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

<u>Rem</u> Subtracting $P(A \cap B)$ compensates for the double counting of the probability of the intersection in $P(A) + P(B)$.

<u>Ex</u> If $P(A) = .5$, $P(B) = .8$, and $P(A \cap B) = .4$, then

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= .5 + .8 - .4 \\ &= .9 \end{aligned}$$

<u>Ex</u> A smoke detector system uses two devizes, a and b. If smoke is present, the probability a detects it is .95 the probability b detects it is .90, and the probability both a and b detect it is .88. (WMS 2.94)

1) Find the probability that smoke will be detected by device a or device b or both.

2) Find the probability that smoke goes undetected.

Let's define the following events.

$A :=$ device a detects smoke

$B :=$ device b detects smoke

Then $A \cup B =$ device a or b or both detect smoke

1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= .95 + .90 - .88 = .97$

Thus, the probability that a or b or both detect smoke is .97

2) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - .97 = .03$

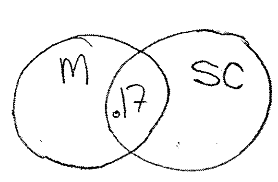
Thus, the probability smoke goes undetected is .03.

Rem What is the sample space rigorously? $\{YY, YN, NY, NN\}$
Find A, B.

Ex 1.2 #2 The experiment consists of randomly selecting a customer from all customers on file, so the outcome space is the set of all customers on file.

$M :=$ set of all customers that insure more than one car

$SC :=$ set of all customers that insure a sports car.



$\overline{M \cup SC} = \overline{M} \cap \overline{SC}$

$P(M) = .85$

$P(SC) = .23$

$S :=$ set of all auto customers

$$P(\overline{M} \cap \overline{SC}) = P(\overline{M \cup SC})$$
$$= 1 - P(M \cup SC)$$
$$= 1 - (P(M) + P(SC) - P(M \cap SC))$$
$$= 1 - (.85 + .23 - .17) = .09$$

<u>Cor</u> (to Additive Law) Let P be a probability measure on a sample space S. Suppose $A, B \subseteq S$ and $A \cap B = \emptyset$. Then $P(A \cup B) = P(A) + P(B)$.

<u>Pr</u> By the Additive Law,

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(\underbrace{A \cap B}_{\overset{\text{by hyp.}}{\emptyset}}) \\
&= P(A) + P(B) - \underbrace{P(\emptyset)}_{0 \text{ by earlier result}} \\
&= P(A) + P(B).
\end{aligned}$$

$\square$

There is also an Additive Law for the probability of a union of 3 events. In fact, there is even a general formula for the probability of a union of n events.

<u>Thm</u> (Additive Law for 3 Events).
Let P be a probability measure on a sample space S. Suppose $A, B, C \subseteq S$. Then

$$\begin{aligned}
P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\
& - P(A \cap B) - P(B \cap C) - P(A \cap C) \\
& + P(A \cap B \cap C).
\end{aligned}$$