

1.3 Combinatorial Tools for Counting

Sample Points

As we have seen, it is important to know the size of a sample space in order to compute probabilities and probability measures. Knowing the size of the sample space and the size of an event (subset) helps us to compute probabilities. Up till now, we have been able to list the elements (e.g. the sample space for doing a sequence of 3 coin flips is $\{HHH, HHT,$
 $HHT, HTT,$
 $TTH, TTT,$
 $THH, THT\}$)

But in many concrete situations it is impossible to list all elements of the sample space by hand. So we need other ways to count. The mathematical area of combinatorics provides us with these other ways to count.

We first review Cartesian products to explain the Basic Principle of Counting (Multiplication Principle). Then we give an overview of the various things we want to count, and finally go into the specifics of each.

Cartesian Products

Def Let A and B be sets. The Cartesian product of A and B is

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$$

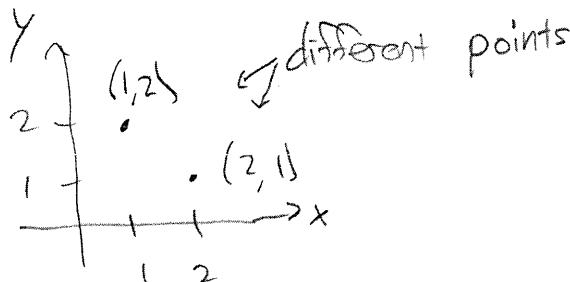
In other words, $A \times B$ is the set of ordered pairs (a, b) with $a \in A$, $b \in B$.

Ex $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

Then $(1, 2) \in \mathbb{R} \times \mathbb{R}$ and $(2, 1) \in \mathbb{R} \times \mathbb{R}$,

and $(1, 2) \neq (2, 1)$,

unlike $\{1, 2\} = \{2, 1\}$.



Ex List the elements of $\{\pi, i\} \times \{3, 4, 5\}$.

$$(\pi, 3)$$

$$(i, 3)$$

$$(\pi, 4)$$

$$(i, 4)$$

$$(\pi, 5)$$

$$(i, 5)$$

Notice: $|\{\pi, i\} \times \{3, 4, 5\}| = 6$

$$= 2 \cdot 3 = |\{\pi, i\}| \cdot |\{3, 4, 5\}|.$$

This motivates the following Thm.

Thm If A and B are finite sets, then
 $|A \times B| = |A| \cdot |B|$. Similarly, if C is also
a finite set, then

$$|A \times B \times C| = |A| \cdot |B| \cdot |C|.$$

Ex Suppose a bus company has 4 daily buses from Dearborn to Lansing, and 5 daily buses from Lansing to Traverse City. Suppose a customer wants to travel from Dearborn to Traverse City, with a one-day break in Lansing. How many different connections are possible?

$$\begin{aligned} A &:= \text{set of bus trips } \text{Dearborn} \rightarrow \text{Lansing} \text{ on day 1} \\ B &:= \text{set of bus trips } \text{Lansing} \rightarrow \text{Traverse City} \text{ on day 3.} \end{aligned}$$

Then $A \times B$ is the set of connections
 $\text{Dearborn} \rightarrow \text{Lansing} \rightarrow \text{Traverse City}$.

$$|A \times B| = |A| \cdot |B| = 4 \cdot 5 = 20.$$

Ex Suppose a restaurant has a fixed price menu with a choice of 3 appetizers, 2 main courses, and 5 deserts. How many different dinners are available? $3 \cdot 2 \cdot 5 = 30$.

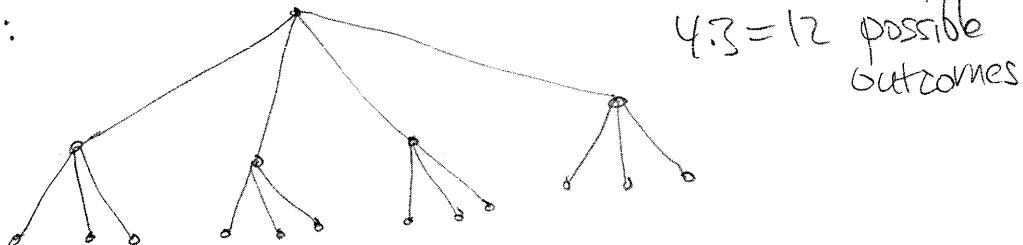
Q. What is a 3-tuple? Why do we consider order?

Ex The sample space of a sequence of 3 coin flips is $\{H, T\} \times \{H, T\} \times \{H, T\}$.

Basic Principle of Counting a.k.a. Multiplication Principle

Basic Principle of Counting: How many possible outcomes are there for two consecutive (or simultaneous) experiments? Suppose there are m possible outcomes for experiment 1 and n possible outcomes for experiment 2. Then all together there are $m \cdot n$ possible outcomes for the composite experiment.

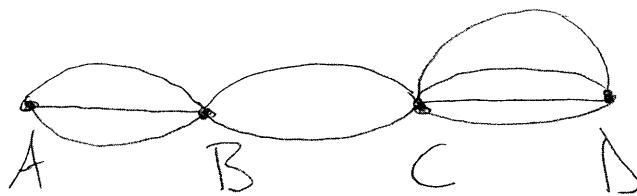
Picture:



Ex Suppose there are 7 baseball teams with 12 players each. Suppose a team, and then a "most valuable player" from that team, will be selected for a prize. How many different choices are possible?

$$7 \cdot 12 = 84.$$

Ex How many ways can I get from town A to town C?



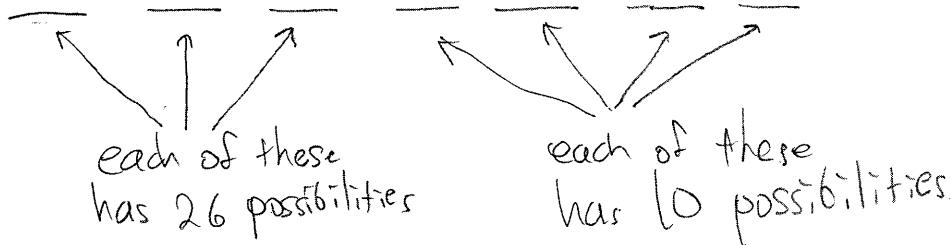
$$3 \cdot 2 = 6$$

From A to D? $3 \cdot 2 \cdot 4 = 24$

Rem A generalization of the basic counting principle also holds when there are k experiments with $n_1, n_2, n_3, \dots, n_k$ outcomes. The composite experiment has $n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_k$ outcomes.

Ex How many 7 digit license plates are possible if the first 3 digits are letters and the next 4 are numbers from 0, 1, ..., 9.

digits:



$$\Rightarrow 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = \boxed{26^3 \cdot 10^4}$$

(Repetitions are allowed here)

Things to Count

Consider the set $\{a, b, c, d, e, f\}$. We would like to count the possibilities of the following things.

- 1) ordered sequences where repeats are allowed
(i.e. sampling with replacement) (use basic counting)
ab, aca, efd, dfe, a, ecfdabbba
- 2) ordered sequences where no repeats are allowed
(i.e. sampling without replacement) (called permutations)
abc, cba, cab, a, defcab (use nPr)
max length, use $nC_r = \binom{n}{r}$
- 3) selections of one unordered subset (combinations)
 $\{a, c, d\} = \{c, d, a\}$, $\{a, d, e, f\}$, $\{d, f\} = \{f, d\}$
- 4) distinguishable ordered sequences of objects
when several objects are indistinguishable.
Ex List all distinguishable sequences of two daffodils and two tulips
TTDD, TDDT, DDTT, TDTD, DTDT, DTTD

1) How to Count Ordered Sequences Where Repeats are Allowed (Sampling With Replacement)

Write down positions, the possibilities for each position, and multiply.

Ex If I have 6 chips in a bowl labelled a,b,c,d,e,f and I successively choose 1 and put it back in four times (sample w/ replacement), how many possible outcomes are there?

$$\underline{6} \quad \underline{6} \quad \underline{6} \quad \underline{6} \Rightarrow 6^4 \quad (\text{Too many to list!})$$

2) How to Count Ordered Sequences Where No Repeats are Allowed (Sampling Without Replacement)

Write down positions, the possibilities for each position, and multiply.

Ex Again 6 labelled chips and 4 successive selections, but I don't put the chips back in after each selection (sample w/o replacement). How many possible outcomes are there?

$$\underline{6} \quad \underline{5} \quad \underline{4} \quad \underline{3} \Rightarrow 6 \cdot 5 \cdot 4 \cdot 3 = \frac{6!}{(6-4)!} = {}_6P_4$$

Such an arrangement is called a permutation of 6 objects taken 4 at a time.

Lemma The number of permutations of n objects taken r at a time is

$$n P_r := \frac{n!}{(n-r)!}$$

Ex Find the number of permutations of $\{a, b, c, d\}$ taken two at a time.

$$\underline{\quad \quad} \quad \underline{\quad \quad} \quad 4 \cdot 3 = 12$$

$$4 P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12$$

can write all twelve :

ab	ba	bc	cb	cd	dc
ac	ca	bd	db		
ad	da				

Def An ordering of n different objects is called a permutation of the n objects.

Ex Consider $\{a, b, c, d\}$. Some permutations are abcd, bacd, cabd, but not aabc, abba, abc.

Lemma There are $n!$ permutations of n different objects.

Ex If you have on a bookshelf 4 topology books, 3 algebra books, and 2 analysis books, how many ways can you arrange them by subject on the bookshelf?

- 4! orderings of topology
- 3! orderings of algebra
- 2! orderings of analysis.

There are $3!$ orderings of the subjects as blocks. The Basic Principle of Counting then implies

$$(3!)(4! \cdot 3! \cdot 2!) \text{ total possibilities.}$$

3) How to Count Selections of One Unordered Subset (Combinations)

Def A combination with r elements from a set is a subset of cardinality r. (Note: a combination is unordered because any subset is unordered.)

Ex List all the combinations with 2 elements from the set $\{a, b, c, d\}$.

$\{a, b\}$	$\{b, c\}$	$\{c, d\}$
$\{a, c\}$	$\{b, d\}$	
$\{a, d\}$		

$$C = \frac{4!}{2!} \cdot \frac{1}{2!} = \frac{4!}{2!(4-2)!}$$

Lemma The number of combinations with r elements selected from a set with n elements is

$${}_n C_r := \binom{n}{r} := \text{"n choose } r\text{"} := \frac{n!}{r!(n-r)!}.$$

Ex Consider a standard deck of 52 cards.

1) How many ways can we select 5 cards from the deck?

$${}_{52} C_5 = \binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!}$$

2) How many ways can we select 5 diamonds from the deck?

$${}_{13} C_5 = \binom{13}{5} = \frac{13!}{5!(13-5)!} = \frac{13!}{5!8!}$$

3) How many ways can we select five cards from the deck, two of which are number cards (Ace - 10) and three of which are face cards (J, Q, K)?

$$\binom{40}{2} \binom{12}{3} = \frac{40!}{2! 38!} \cdot \frac{12!}{3! 9!}$$

by the Basic Principle of Counting

(Simultaneously select two cards from the 40 number cards, and three cards from the 12 face cards)

Ex Consider a bowl with 5 chips labelled a,b,c,d,e. Consider the number of outcomes of the following two experiments.

1) simultaneously select 3 chips (i.e. select a combination of 3)

2) take one chip, then another, and then another, (i.e. select a permutation taken 3 at a time)

then put all three in your bag, and forget the order, so in the end you only have a combination.

Q. Are the outcomes the same? Do we have the same number of outcomes?

Yes to both!

1) There are ${}^5C_3 = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!}$

possible outcomes

2) In the first step of 2), where we select a permutation, there are ${}^5P_3 = \frac{5!}{(5-3)!}$ possible permutations. But then we forget about the order and consider $6=3!$ permutations of the same three letters as the same; for instance

abc	bca	cab
acb	cba	bac

all become $\{a, b, c\}$.

So we divide the permutation total the first step by $3!$ to get

$$\frac{5!}{(5-3)!}/3! = \frac{5!}{2!3!} = \text{Same Number as in situation 1.}$$

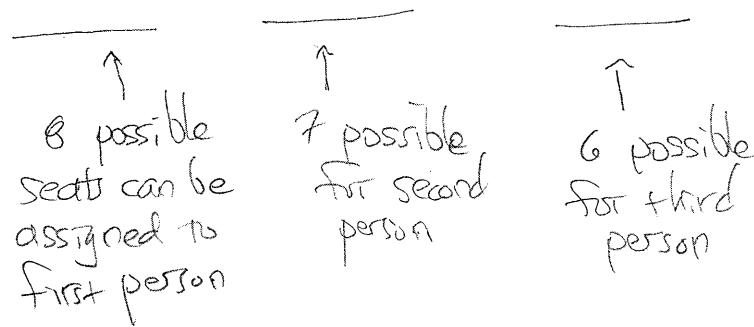
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So, we see that choosing three simultaneously is the same as choosing three successively and forgetting the order.

Ex How many ways are there to seat 3 people in a row of 8 chairs?

This can be solved using nPr or nCr !

Method 1: Imagine the people as locations for numbered chairs.



$$\Rightarrow 8 \cdot 7 \cdot 6 \text{ possible ways to seat 3 people in 8 chairs, i.e. } 8P_3 = \frac{8!}{(8-3)!}$$

Method 2: First select a subset of 3 chairs for people to sit in, then seat the 3 people in those 3 chairs. Count the possibilities of each step and multiply.

$${}_8C_3 \cdot 3! = \binom{8}{3} \cdot 3! = \frac{8!}{3!(8-3)!} \cdot 3!$$

$$= \frac{8!}{5!} = 8 \cdot 7 \cdot 6$$

Same as above!

45 How to Count Distinguishable Ordered Sequences of Objects when Several Objects are Indistinguishable

Ex List all distinguishable sequences of two daffodils and two tulips.

TTDD, TDST, DSTT, TDTS, STDT, STTD

Rem Notice that each sequence is determined by selecting which positions will get tulips (then the rest get daffodils). Thus selecting an unordered set of two positions from 4 is $\binom{4}{2} = \frac{4!}{2!2!} = 6$, as above.

Ex Consider all sequences of 15 coin flips. How many have exactly 7 heads and 8 tails?

$$\binom{15}{7} = \frac{15!}{7!8!}$$

(Select as selecting 7 positions from 15)

Thm Suppose we have n objects, divided up into k similarity groups. Suppose the first group has n_1 similar objects, the second has n_2 similar objects, etc. Then the number of distinguishable ordered sequences is the multinomial coefficient

$$\binom{n}{n_1 \ n_2 \ \dots \ n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Rem Notice $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{r \ n-r}$.

Thus in the flower example we have

$$\binom{4}{2 \ 2} = \frac{4!}{2!2!} = 6$$

and to the coin example we have

$$\binom{15}{7 \ 8} = \frac{15!}{7!8!}$$

Ex If we have 5 roses, 2 tulips, and 7 orchids, how many distinguishable sequences are there?

$$\binom{14}{5 \ 2 \ 7} = \frac{14!}{5!2!7!}$$

Ex If a restaurant has 14 employees, and 2 people are needed to manage take-out, 5 are needed to wait tables, 4 are needed to cook, and 3 are needed to wash dishes, how many ways can the jobs be assigned?

A. Think of the people as positions rather than the jobs!

Divide up the jobs into similarity groups, and arrange the people in a line. Putting the jobs in a sequence is the same as assigning them to people. \Rightarrow Now we can apply the Theorem.

$$\binom{14}{2 \ 5 \ 4 \ 3} = \frac{14!}{2! \cdot 5! \cdot 4! \cdot 3!}$$

Further Elaboration on Type 3 Example 3:

How to Count Subsets in which a Specified Number of Elements in the Subset have Certain Characteristics

This topic is helpful on several problems in WebWork 2 and will be important later for several kinds of discrete random variables.

Ex A bag contains 7 red marbles, 8 white marbles, and 9 blue marbles. You simultaneously draw 4 marbles out at random.

a) What is the probability all 4 are red?

$$\text{probability} = \frac{\text{number of ways to draw 4 reds}}{\text{number of ways to draw any 4}}$$

$$= \frac{\binom{7}{4}}{\binom{24}{4}} = \frac{\frac{7!}{4!3!}}{\frac{24!}{4!20!}}$$

b) What is the probability 2 are red and 2 are white?

(We use the Basic Principle of Counting.

$$\text{probability} = \frac{\left(\begin{array}{l} \text{number of ways} \\ \text{to select 2 reds} \end{array} \right) \cdot \left(\begin{array}{l} \text{number of ways} \\ \text{to select 2 whites} \end{array} \right)}{\left(\begin{array}{l} \text{number of ways} \\ \text{to select any 4} \end{array} \right)}$$

$$= \frac{\binom{7}{2} \cdot \binom{8}{2}}{\binom{24}{4}} = \frac{\frac{7!}{2!5!} \cdot \frac{8!}{2!6!}}{\frac{24!}{4!20!}}$$

c) What is the probability 2 are red, 1 is white, and 1 is blue?

Again use the Basic Principle of Counting.

$$\text{probability} = \frac{\left(\begin{array}{l} \text{number of ways} \\ \text{to select 2 reds} \end{array} \right) \cdot \left(\begin{array}{l} \text{number of ways} \\ \text{to select 1 white} \end{array} \right) \cdot \left(\begin{array}{l} \text{number of ways} \\ \text{to select 1 blue} \end{array} \right)}{\left(\begin{array}{l} \text{number of ways} \\ \text{to select any 4} \end{array} \right)}$$

$$= \frac{\binom{7}{2} \cdot \binom{8}{1} \cdot \binom{9}{1}}{\binom{24}{4}}$$

d) What is the probability exactly 2 are red?

Again use the Basic Principle of Counting,
but consider only two groups: red and
non-red.

$$\text{probability} = \frac{\left(\begin{array}{l} \text{number of ways} \\ \text{to select 2 reds} \end{array} \right) \cdot \left(\begin{array}{l} \text{number of ways} \\ \text{to select 2 non-reds} \end{array} \right)}{\left(\begin{array}{l} \text{number of ways} \\ \text{to select any 4} \end{array} \right)}$$

$$= \frac{\binom{7}{2} \cdot \binom{17}{2}}{\binom{24}{4}}$$

e) What is the probability at least 1 is red?

Use the Complement Law.

$$\text{probability} = 1 - \left(\begin{array}{l} \text{probability of} \\ \text{selecting 0 reds} \end{array} \right)$$

$$= 1 - \left(\begin{array}{l} \text{probability of} \\ \text{selecting 4 non-reds} \end{array} \right)$$

$$= 1 - \frac{\left(\begin{array}{l} \text{number of ways} \\ \text{to select 4 non-reds} \end{array} \right)}{\left(\begin{array}{l} \text{number of ways} \\ \text{to select any 4} \end{array} \right)}$$

$$= 1 - \frac{\binom{17}{4}}{\binom{24}{4}}$$