1.5 Independent Events

When the occurence of an event A is unaffected by the occurence or non-occurence of an event B, we say they are independent.

Def Two events $A, B \subseteq S$ are independent if $P(A \cap B) = P(A) \cdot P(B)$.
The events A and B are dependent if $P(A \cap B) \neq P(A) \cdot P(B)$.

Ex A coin is flipped twice. Let

$A :=$ event that first flip is heads
$B :=$ event that second flip is heads

Are A and B independent? Let's check, using the definition.

$$S := \{HH, HT, TH, TT\}$$

$$A := \{HH, HT\} \qquad \Rightarrow A \cap B = \{HH\}$$
$$B := \{HH, TH\}$$

$$P(A \cap B) = \tfrac{1}{4} \qquad P(A) \cdot P(B) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

So $P(A \cap B) = P(A) \cdot P(B) \Rightarrow A, B$ are independent.

Ex A bowl has 10 chips labelled 1 through 10. We sample twice with replacement, i.e. the outcome is a sequence of two numbers where repeats are allowed.

$A :=$ the event that the first draw is even

$B :=$ the event that the second draw is even.

Are A and B independent?

$|A| = 5 \cdot 10 = 50 \qquad |B| = 10 \cdot 5 = 50$

$|A \cap B| = 5 \cdot 5 = 25$

$$P(A \cap B) = \frac{25}{100} \qquad P(A) \cdot P(B) = \underbrace{\frac{50}{100} \cdot \frac{50}{100}}_{\text{or think of it as } \frac{5}{10} \cdot \frac{5}{10}} = \frac{25}{100}$$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow$ A and B are <u>independent</u>.

Ex Again consider a bowl with 10 chips labelled 1 through 10, but this time we sample twice <u>without</u> replacement, i.e. the outcome is a sequence of two numbers where repeats are not allowed. Again we consider the events

$A :=$ the event that the first draw is even

$B :=$ the event that the second draw is even

Are A and B independent?

$P(A \cap B) = P(A)P(B|A)$ by the Multiplicative Law

$$= \underbrace{\frac{5}{10} \cdot \frac{4}{9}} = \frac{20}{90} = .\overline{2}$$

when the first chip is one of the 5 evens, then only 4 of the remaining 9 are even.

$$P(A) = \frac{5}{10}$$

$P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A})$ by the Law of Total Probability

$$= \frac{4}{9} \cdot \frac{5}{10} + \underbrace{\frac{5}{9}} \cdot \frac{5}{10}$$

when the first chip is not one of 5 evens, then 5 of the remaining 9 are even.

$$= \frac{45}{90}$$

$$P(A) \cdot P(B) = \frac{5}{10} \cdot \frac{45}{90} = \frac{1}{4} = .25 \overset{?}{\neq} \Rightarrow P(A \cap B) \neq P(A)P(B)$$

$\Rightarrow$ A and B are <u>dependent</u>

<u>Prop</u> Let $A, B \subseteq S$ be events for some experiment.

1) Suppose $P(A) \neq 0$. Then A and B are independent if and only if

$$P(B|A) = P(B).$$

2) Suppose $P(B) \neq 0$. Then A and B are independent if and only if

$$P(A|B) = P(A).$$

<u>Pr</u> 1) A and B independent $\overset{\text{Def}}{\iff} P(A \cap B) = P(A) \cdot P(B)$

$$\overset{P(A)\neq 0}{\iff} \frac{P(A \cap B)}{P(A)} = P(B)$$

$$\overset{A\cap B = B\cap A}{\iff} \frac{P(B \cap A)}{P(A)} = P(B)$$

$$\overset{\text{Def}}{\iff} P(B|A) = P(B). \checkmark$$

2) Similar.

$\square$

Exercise In the examples above, compute $P(A|B)$ and $P(A)$ and compare them in light of the proposition. Notice: an equality $P(A|B) = P(A)$ means that the occurence of B has no effect on the probability of A.

Thm If A and B are independent events then

1) A and $\overline{B}$ are independent
2) $\overline{A}$ and B are independent
3) $\overline{A}$ and $\overline{B}$ are independent.

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Relationship Between Mutually Exclusive Events and Dependence

Prop If A and B are mutually exclusive events with nonzero probability, then A and B are dependent.

Pr $P(A \cap B) = P(\emptyset) = 0$

$P(A) \cdot P(B) \neq 0$ because $P(A) \neq 0$ and $P(B) \neq 0$.

$\Rightarrow P(A \cap B) \neq P(A) \cdot P(B) \Rightarrow A$ and B are dependent.

$\square$

Ex Roll of die, $A := \{2\}$, $B = \{1,3,5\}$, $P(A) = \frac{1}{6}$, $P(A|B) = 0$.

Rem The converse to the Proposition is not true, i.e. A and B dependent $\not\Rightarrow$ A and B mutually exclusive

For a counterexample, look at the example above about selecting twice without replacement from a bowl of 10 chips. There A and B are dependent, and $(2,4) \in A \cap B$, so $A \cap B \neq \emptyset$.

Rem We just saw that dependence does not imply mutual exclusiveness. What about independence, does independence imply mutual exclusiveness? No.

$$A \text{ and } B \text{ independent} \not\Rightarrow A \text{ and } B \text{ mutually exclusive.}$$

For a counterexample, look at the example of the twice flipped coin above. There A and B are independent, and

$$A \cap B = \{HH\} \neq \emptyset.$$

So now we understand the relationship between mutual exclusiveness and dependence, and saw that independence does not imply mutual exclusiveness.

In a different direction, we have the following:

<u>Prop</u> Let $A, B \subseteq S$ be events, not necessarily mutually exclusive.
If $P(A) = 0$, then A and B are independent.

<u>Proof</u>

$$\emptyset \subseteq A \cap B \subseteq A$$

$$\underset{\substack{\| \\ 0}}{P(\emptyset)} \le P(A \cap B) \le \underset{\substack{\| \\ 0}}{P(A)}$$

$$\Rightarrow \quad P(A \cap B) = 0.$$

$$\Rightarrow \quad P(A \cap B) = P(A) \cdot P(B). \checkmark$$

$\square$