1.6 Bayes' Theorem

We saw earlier that $P(A|B)$ is not necessarily equal to $P(B|A)$. So how do you find $P(B|A)$ when you know $P(A|B)$? We now learn two results on finding $P(B|A)$ from $P(A|B)$. The second result is Bayes' Theorem, and is quite important.

<u>Thm</u> Let A and B be events with nonzero probability in a sample space for some experiment. Then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

<u>Pr</u> Multiplicative Law $\Rightarrow P(B|A)P(A) = P(A\cap B) = P(A|B)P(B)$

$$\Rightarrow P(B|A)P(A) = P(A|B)P(B)$$

$$\Rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$\square$

Ex (WeBWorK 3 #4) In a certain community, 32% of the families own a dog, and 25% of the families that own a dog also own a cat. It is also known that 28% of all the families own a cat.

What is the probability that a randomly selected family owns a dog?

Events $D :=$ own a dog

$C :=$ own a cat

$P(D) = .32 \qquad P(C|D) = .25 \qquad P(C) = .28$

$P(D) = .32$ answers the question.

What is the conditional probability that a randomly selected family owns a dog given that it doesn't own a cat?

We're interested in $P(D|\overline{C})$.

$$P(D|\bar{C}) = \frac{P(\bar{C}|D)P(D)}{P(\bar{C})}$$

$$= \frac{(1-P(C|D))P(D)}{1-P(C)}$$

$$= \frac{(1-.25)\cdot .32}{1-.28}$$

$$= .\bar{3}$$

<u>Thm</u> (Bayes' Thm) Let A and B be events with nonzero probability in a sample space for some experiment. Suppose also $\bar{B}$ has nonzero probability. Then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

<u>Pr</u> $P(B|A) \underset{\substack{\uparrow \\ \text{previous} \\ \text{Thm}}}{=} \dfrac{P(A|B)P(B)}{P(A)} \underset{\substack{\uparrow \\ \text{Law of Total Probability} \\ \text{in denominator}}}{=} \dfrac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$

$\square$

Ex (WeBWork 3 #9)

You ask a neighbor to water a sickly plant while you are on vacation. Without water the plant will die with probability .9. With water it will die with probability .6. You are 82% certain the neighbor will remember to water the plant.

1) Find the probability the plant will die while you are on vacation.

Events $D :=$ plant dies while on vacation
$W :=$ plant is watered while on vacation

$P(D|\overline{W}) = .9$

$P(D|W) = .6$

$P(W) = .82$

$$P(D) = P(D|W) \cdot P(W) + P(D|\overline{W}) P(\overline{W})$$
$$= .6 \cdot .82 + .9 \cdot (1 - .82)$$
$$= .654$$

2) You come back from vacation and the plant is dead. What is the probability the neighbor forgot to water it?

$$P(\overline{W}|D) \overset{\text{Bayes' Thm}}{=} \frac{P(D|\overline{W})P(\overline{W})}{P(D|W)P(W)+P(D|\overline{W})P(\overline{W})}$$

$$= \frac{P(D|\overline{W})P(\overline{W})}{P(D)}$$ already found denom!

$$= \frac{.9\cdot(1-.82)}{.654}$$

$$= .2477$$

Bayes' Theorem and Diagnostic Testing

Consider a diagnostic test for a disease. The outcome of the test is either + or −. Of course, the test does not always give the true answer. The test may give <u>false positives</u> sometimes, i.e. the test says + although the person does not have the disease. Or, the test may sometimes give <u>false negatives</u>, i.e. the test says − although the person actually has the disease.

<u>Ex</u> Suppose the following about a blood test for a fictional disease.

- If a person has the disease, the test says + with .95 probability.
- If a person does <u>not</u> have the disease, the test says + with .01 probability.
- .5% of the population has the disease.

What is the probability a person has the disease if the person's test is +?

$D :=$ event the tested person has the disease

$+ :=$ event the person's test result is $+$

$$P(D|+) \overset{\text{Bayes' Thm}}{=} \frac{P(+|D)\cdot P(D)}{P(+|D)P(D) + P(+|\bar{D})P(\bar{D})}$$

$$= \frac{(.95)\cdot(.005)}{(.95)(.005) + (.01)(.995)}$$

$$\approx .323$$

Thus, only 32% of the people who test positive actually have the disease!

<u>Def</u> <u>sensitivity of test</u> $:= P(+|D)$

<u>specificity of test</u> $:= P(-|\bar{D})$

<u>positive predictive value of test</u> $:= P(D|+)$

Bayes' Theorem and Additional Information

Bayes' Theorem is also good for improving probabilities when new information is gained.

Ex (Criminal Investigation) Suppose a detective is 60% sure Suspect A committed the crime. Then, new evidence comes to light that proves whoever committed the crime is left handed. Only 10% of the population is left handed. If Suspect A turns out to be left handed, what is the detective's revised certainty that Suspect A committed the crime?

$C :=$ event that suspect committed crime
$L :=$ event that suspect is left handed

$$P(C|L) = \frac{P(L|C)P(C)}{P(L|C)P(C) + P(L|\bar{C})P(\bar{C})}$$

$$= \frac{1\cdot(.60)}{1\cdot(.60) + (.10)(.40)} = .80$$

$\Rightarrow$ Detective is now 80% sure!

Generalized Bayes Theorem

We stated the Law of Total Probability and Bayes' Theorem for partitions of S into two subsets: $S = B \cup \bar{B}$.

disjoint

But there is also a version of each for partitions of S into m subsets.

Generalized Law of Total Probability Let $B_1, \ldots, B_m$ be a partition of S. That is, $S = \bigcup_{i=1}^{m} B_i$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$. Suppose each B_i has positive probability. Let $A \subseteq S$. Then $P(A) = \sum_{i=1}^{m} P(A|B_i) \cdot P(B_i)$.

Generalized Bayes' Theorem Let $B_1, \ldots, B_k$ be a partition of S, and $A \subseteq S$. Suppose A and all B_i's have positive probability.

Then $P(B_k|A) = \dfrac{P(A|B_k) P(B_k)}{\sum_{i=1}^{m} P(A|B_i) P(B_i)}$ for each $1 \leq k \leq m$.

The Generalized Bayes' Theorem is needed on #11 of WeBWorK 3.