2.2 The Expected Value of a Discrete Random Variable and the Expected Value of a Function of a Discrete Random Variable

Def Let Y be a discrete random variable. The expected value of Y or mean of Y is $E(Y) := \sum_y y f(y)$ where f is the probability mass function of Y. The sum is over all y in $\text{Image}(Y)$ such that $f(y) > 0$.

Ex 2) (From last time) Consider the experiment of tossing a fair coin 3 times and recording the sequence of heads and tails. Let $Y(s)$ be the number of heads in the sequence s.

$$E(Y) = \sum_y y f(y) = 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3)$$
$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8}$$
$$= 1.5$$

The frequentist interpretation of $E(Y)=1.5$ is: if we repeatedly toss a fair coin 3 times and count the number of heads, say 1,000 times, or a million times, then on average there will be approximately 1.5 heads in each sequence of 3.

Ex 3) Consider the experiment of rolling a fair die and recording the outcome. Let $Y: S \to \mathbb{R}$ be $Y(s)=s$. In other words, the random variable Y just outputs the outcome. Then

$$E(Y) = \sum_y y f(y) = 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) + 4 \cdot f(4) + 5 \cdot f(5) + 6 \cdot f(6)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

Notice: $E(Y)$ in this case is the average of the outcomes because each sample point is equally likely.

The frequentist interpretation of $E(Y) = 3.5$ is: if we roll a die many times, then on average the outcome will be approximately 3.5.

Ex (Average Earnings of a Repeated Bet). Suppose you have a coin which is heads with probability $\frac{1}{4}$. If heads is flipped, your friend gives you \$4. If tails is flipped, you give your friend \$1. Repeat many times. What are your average winnings per game?

Let $Y: \{H, T\} \to \mathbb{R}$ be
$$Y(H) := 4$$
$$Y(T) := -1$$

Then its mass function is $f: \{4, -1\} \to [0,1]$

$$f(4) = P(Y=4) = P(\{H\}) = \frac{1}{4}$$
$$f(-1) = P(Y=-1) = P(\{T\}) = \frac{3}{4}.$$

This p.m.f. means 25% of the time we win \$4, and 75% of the time we lose \$1.

$$E(Y) = \sum_y y f(y) = 4 \cdot f(4) + (-1) \cdot f(-1)$$

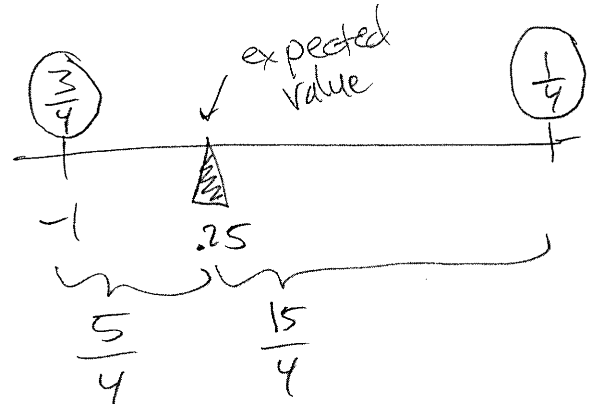
$$= 4 \cdot \frac{1}{4} + (-1) \cdot \frac{3}{4}$$

$$= \$.25$$

$\Rightarrow$ Your average earnings are 25 cents per game.

Rem 1) The expected value of a random variable is a weighted average of the possible values of the random variable. The weights are the respective probabilities of the values of the random variable. (See the above examples!)

<u>Rem</u> 2) The expected value of a random variable is the "center of mass" for weights $f(y)$ placed at y. For instance, in the bet example from above



$E(Y) = .25$

is balanced according to the Law of the Lever:

$$\frac{3}{4} \cdot \frac{5}{4} = \frac{1}{4} \cdot \frac{15}{4} \ . \ \checkmark$$

The probability histogram would balance on $E(Y) = .25$. $E(Y) = .25$ is the "center of mass" of the probability histogram.

<u>Rem</u> 3) When each sample point is equally likely, the expected value $E(Y)$ is the mean of the values of the random variable (see the fair die example 3) above).

The Expected Value of a Function of a Discrete Random Variable

Thm Let Y and Z be discrete random variables on the same sample space S. Let c be a real constant. Then

i) $E(c) = c$

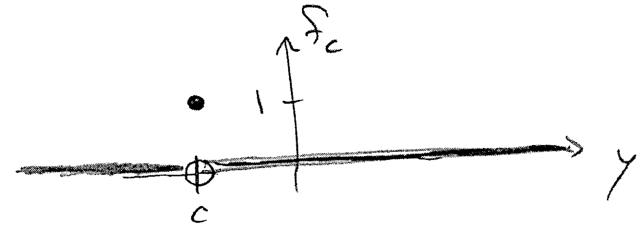
ii) $E(cY) = cE(Y)$

iii) $E(Y+Z) = E(Y) + E(Z)$

$\Big\}$ These two together say E is "linear."

Pr i) Let $f_c : \mathbb{R} \longrightarrow [0,1]$ be the p.m.f. of the constant c random variable $c : S \longrightarrow \mathbb{R}$, $s \longmapsto c$.

Then $f_c(y) = P(c=y) = \begin{cases} 1 & \text{if } y=c \\ 0 & \text{if } y \neq c. \end{cases}$



$$E(c) = \sum_y y f_c(y) = \sum_{\substack{y \text{ st.} \\ f_c(y)>0}} y f_c(y) = c f_c(c) = c \cdot 1 = c$$

✓

ii) Let $f_{cY} : \text{image}(cY) \to [0,1]$ be the probability mass function of the discrete random variable $cY : S \to \mathbb{R}$

$$s \mapsto c \cdot Y(s).$$

Then $f_{cY}(y) \overset{\text{def}}{=} P(cY = y)$

$= P(Y = \frac{y}{c})$ because $(cY = y) = (Y = \frac{y}{c})$

$\overset{\text{def}}{=} f_Y(\frac{y}{c})$ where f_Y is the pmf of Y.

$$\Rightarrow E(cY) = \sum_y y f_{cY}(y)$$

$\overset{\text{above}}{=} \sum_y y f_Y(\frac{y}{c})$ Let $y' = \frac{y}{c}$ for change of variable

$= \sum_{y'} cy' f_Y(y')$

$= c\left(\sum_{y'} y' f_Y(y')\right) = cE(Y)$ ✓

iii) Let $Y+Z : S \longrightarrow \mathbb{R}$ be the random variable defined by $(Y+Z)(s) := Y(s) + Z(s)$ for all $s \in S$.

To prove $E(Y+Z) = E(Y) + E(Z)$, one uses <u>Thm</u> For Y a discrete random variable on S,

$$E(Y) = \sum_{s \in S} Y(s) \cdot P(\{s\}).$$

We leave it as an exercise. ✓

$\square$

To find the expected value of a more general function of a discrete random variable, we have the following.

Thm Let Y be a discrete random variable and $g: \text{image}(Y) \to \mathbb{R}$ any function. Let $g(Y)$ denote the random variable $g \circ Y$. Then

$$E(g(Y)) = \sum_y g(y) f(y).$$

Ex Suppose

y	$f(y)$
1	1/5
2	2/5
3	2/5

is the p.m.f. for a discrete random variable Y.

Find $E(Y)$, $E(Y^2)$, and $E(1/Y)$.

$$E(Y) = \sum_y y f(y) = 1 \cdot \frac{1}{5} + 2 \cdot \frac{2}{5} + 3 \cdot \frac{2}{5} = \frac{11}{5}$$

$$E(Y^2) = \sum_y y^2 f(y) = 1^2 \cdot \frac{1}{5} + 2^2 \cdot \frac{2}{5} + 3^2 \cdot \frac{2}{5} = \frac{27}{5}$$

$$E(1/Y) = \sum_y \frac{1}{y} f(y) = \frac{1}{1} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{2}{5} = \frac{8}{15}$$