2.4 Binomial Random Variables

Motivating Example 1: "The flip of a coin is a Bernoulli experiment."

Suppose an experiment is the flip of an (unfair) coin for which heads has probability $p = .8$.
This experiment has exactly two mutually exclusive outcomes: heads or tails. The sample space is

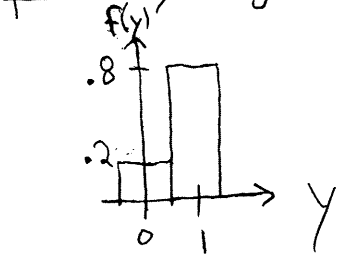
$$S = \{H, T\}.$$

Consider the random variable $Y : S \to \mathbb{R}$ given by $Y(s) := \begin{cases} 1 & \text{if } s = H \\ 0 & \text{if } s = T \end{cases}$.

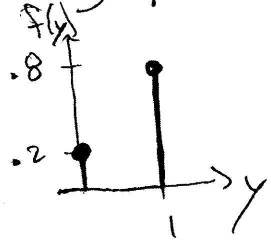
Y is called a Bernoulli random variable. Its p.m.f. is called a Bernoulli distribution:

$f : \{0,1\} \to [0,1]$ is
$$f(0) = P(Y=0) = P(\{T\}) = \overbrace{1-p}^{q} = .2$$
$$f(1) = P(Y=1) = P(\{H\}) = p = .8$$

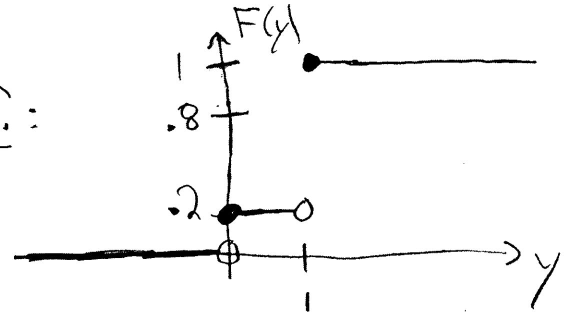
probability histogram:



bar graph:



c.d.f.:



The p.m.f. $\begin{array}{c|c|c} y & 0 & 1 \\ \hline f(y) & .2 & .8 \end{array}$ can be given

by the formula $f(y) = .8^y \cdot .2^{(1-y)}$

or $f(y) = p^y (1-p)^{1-y}$.

try it: $f(0) = p^0 (1-p)^1 = 1-p$ ✓

$f(1) = p^1 (1-p)^{1-1} = p$ ✓.

The <u>expected value</u> of a Bernoulli random variable Y is

$$\mu = E(Y) \overset{\text{def}}{=} \sum_{y \in \text{range}(Y)} y \cdot f(y) = 0 \cdot f(0) + 1 \cdot f(1) = p.$$

so $\boxed{\mu = p}$.

The <u>variance</u> of a Bernoulli random variable Y is

$$\sigma^2 \overset{\text{def}}{=} E\left((Y-\mu)^2\right) \overset{\text{thm}}{=} \sum_{y \in \text{range}(Y)} (y-\mu)^2 \cdot f(y)$$

$$= (0-p)^2 \cdot f(0) + (1-p)^2 f(1)$$

$$= p^2 \cdot (1-p) + (1-p)^2 p$$ factor out $(1-p)$

$$= (1-p)(p^2 + (1-p)p)$$

$$= (1-p)(p^2 + p - p^2)$$

$$= (1-p)p$$

Thus $\boxed{\sigma^2 = p(1-p)}$.

<u>Motivating Example 2</u>: "A sequence of 3 coin flips is a <u>binomial</u> experiment."

Suppose an experiment consists of a sequence of 3 flips of an (unfair) coin for which heads has probability $p = .8$. The sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Consider the random variable $X : S \to \mathbb{R}$ given by $X(s) :=$ the number of H's in the sequence s.

e.g. $X(HHH) = 3$ and $X(TTH) = 1$.

X is an example of a <u>binomial random variable.</u>

Notice that X is the sum of 3 Bernoulli random variables: Let $Y_1, Y_2, Y_3 : S \to \mathbb{R}$ be defined by

$$Y_i(s) := \begin{cases} 1 & \text{if the i-th letter in s is H} \\ 0 & \text{if the i-th letter in s is T} \end{cases}.$$

Then $X(s) = Y_1(s) + Y_2(s) + Y_3(s)$ for all $s \in S$, so $X = Y_1 + Y_2 + Y_3$ as random variables.

Let's compute the p.m.f. of X, $f: \underbrace{\{0,1,2,3\}}_{=\text{range}(X)} \longrightarrow [0,1]$.

$f(0) = P(X=0) = P(\{TTT\}) = q \cdot q \cdot q = q^3$ where $q := 1-p$.

(the flips are identical and independent experiments, so we multiply $q \cdot q \cdot q$ to find the probability of TTT).

$f(1) = P(X=1) = P(\{HTT, THT, TTH\})$

$= pqq + qpq + qqp$

$= 3 \cdot pq^2$

$= \begin{pmatrix} \text{number of ways to flip} \\ \text{exactly 1 head in 3 flips} \end{pmatrix} \cdot pq^2$

$= \binom{3}{1} pq^2$

$= \binom{3}{1} p^1 \cdot q^{3-1}$

Recall $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ = number of ways of selecting k things from n

Similarly,

$f(2) = P(X=2) = P(\{HHT, HTH, THH\})$

$= ppq + pqp + qpp$

$= 3p^2q = \begin{pmatrix} \text{number of ways to flip} \\ \text{exactly 2 heads in 3 flips} \end{pmatrix} \cdot p^2 q$

$= \binom{3}{2} p^2 q$

$= \binom{3}{2} p^2 q^{3-2}$

$$f(3) = P(X=3) = P(\{HHH\})$$
$$= ppp$$
$$= p^3$$

Thus, we observe the pattern $\boxed{f(x) = \binom{3}{x} p^x q^{3-x}}$.

a) What is the probability of exactly two heads?

$$P(X=2) = f(2) = \binom{3}{2} p^2 q^{3-2}$$
$$= \binom{3}{2} .8^2 .2^1 = .384$$
$$= \text{dbinom}(\underset{\substack{\uparrow \\ \text{\# of} \\ \text{heads}}}{2}, \underset{\substack{\uparrow \\ \text{\# of} \\ \text{trials}}}{3}, \underset{\substack{\uparrow \\ \text{success} \\ \text{probability}}}{.8}) \text{ in R}$$

b) What is the probability of at most two heads?

$$P(X \le 2) = P(\underbrace{(X=0) \cup (X=1) \cup (X=2)}_{\text{pairwise mutually exclusive events}})$$
$$= P(X=0) + P(X=1) + P(X=2)$$
$$= f(0) + f(1) + f(2)$$
$$= \binom{3}{0} p^0 q^3 + \binom{3}{1} p^1 q^2 + \binom{3}{2} p^2 q^1$$
$$= \binom{3}{0} .8^0 .2^3 + \binom{3}{1} .8 .2^2 + \binom{3}{2} .8^2 .2^1 = \underset{\text{= c.d.f. at 2}}{\overset{\substack{.488 \\ \|}}{\text{pbinom}(2,3,.8)}}$$

c) What is the probability of less than 2 heads?

$$P(X<2) = P(\underbrace{(X=0) \cup (X=1)}_{\text{mutually exclusive events}})$$

$$= P(X=0) + P(X=1)$$

$$= f(0) + f(1)$$

$$= \binom{3}{0} p^0 q^3 + \binom{3}{1} p^1 q^2$$

$$= \binom{3}{0} .8^0 .2^3 + \binom{3}{1} .8\ .2^2$$

$$= \text{pbinom}(1,3,.8) = .104$$

Notice that $P(X \le 2) \ne P(X<2)$, and that $\le$ versus $<$ makes a difference for discrete random variables (unlike for continuous random variables).

We have just seen a concrete example of a binomial experiment, a binomial random variable, and a binomial distribution. Let's turn to the general definition.

Def A <u>binomial experiment</u> is an experiment for which the following are true.

1. The experiment consists of a fixed number, say n, of identical trials.

2. Each trial results in exactly one of two outcomes: success (S) or failure (F).

3. The probability of success on any of the single trials is p, no matter which trial. The probability of failure on any of the single trials is $q := 1-p$.

4. The trials are independent.

In this situation, the sample space is

$$\{(a_1, a_2, a_3, .., a_n) \mid a_i \in \{S, F\} \text{ for all } i = 1, .., n\}$$

and the random variable X on this sample space defined by $X((a_1, a_2, .., a_n)) =$ the number of successes in this sequence $(a_1, .., a_n)$ is called a <u>binomial random variable</u>.

Lemma Suppose X is a binomial random variable with n trials and success probability p. Then its p.m.f. is $f : \{0, 1, \ldots, n\} \to [0, 1]$

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

where $q := 1 - p$.

The c.d.f. of X is $F : \mathbb{R} \to \mathbb{R}$

$$F(x) = \sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k q^{n-k}$$

where $\lfloor x \rfloor :=$ the greatest integer less than or equal to x, and $\binom{n}{k} = 0$ when $k > n$ by convention.

Thm Suppose X is a binomial random variable with n trials and success probability p. Then its expected value is $E(X) = np$ and its variance is $\mathrm{Var}(X) = npq$.

Rem Success and failure can be replaced by any two mutually exclusive outcomes in the definition, and we still call it a binomial random variable.

Notation The notation "X is $b(n,p)$" means X is a binomial random variable with n trials and success probability p.