2.5 The Moment Generating Function of a Discrete Random Variable

Motivation Recall from algebra that two polynomials are equal if and only if their coefficients are equal. More generally, recall from Calculus 2 that two convergent power series are equal if and only if their coefficients are equal. Moreover, the nth coefficient of a power series about 0 for $g(t)$ is $\dfrac{g^{(n)}(0)}{n!}$.

We will similarly see that two discrete random variables X, Y have the same p.m.f. if and only if the two power series with coefficients

$$\frac{E(X^n)}{n!} \quad \text{and} \quad \frac{E(Y^n)}{n!} \quad \text{are equal}$$

(provided they converge).

There is a long tradition in math of encoding information in the coefficients of power series (see "Generating Functionology").

The moment generating function encodes the $E(X^n)$ (called "moments about 0") and the p.m.f. of X, so continues this tradition.

<u>Further Motivation</u> We will need moment generating functions in Section 5.4 where we learn how to compute the pmf/pdf of a linear combination of random variables with known pmf's / pdf's.

<u>The Moment Generating Function of a Discrete Random Variable</u>

<u>Def</u> Let X be a discrete random variable with pmf $f(x)$. The <u>moment generating function of X</u> is

$$M(t) := E(e^{tX}) \overset{\text{Thm}}{=} f(x_1)e^{x_1 t} + f(x_2)e^{x_2 t} + f(x_3)e^{x_3 t} + \cdots + f(x_n)e^{x_n t} + \cdots$$

where $\text{image}(X) = \text{supp}(X) = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$.

Def The moment generating function

$$M(t) = \sum_x f(x) e^{tx}$$ is said to exist

if the sum converges for all t in $(-h, h)$ for some positive real h.

Ex If the p.m.f. of X is

x	5	7	10
$f(x)$	1/9	3/9	5/9

then the m.g.f. of X is

$$M(t) = \frac{1}{9}e^{5t} + \frac{3}{9}e^{7t} + \frac{5}{9}e^{10t}.$$

Ex If the m.g.f. of X is $M(t) = \frac{4}{7}e^{10t} + \frac{1}{7}e^{20t} + \frac{2}{7}e^{45t}$, then the p.m.f. of X is

x	10	20	45
$f(x)$	4/7	1/7	2/7

In fact, we can always move back and forth between p.m.f.'s and m.g.f.'s.

<u>Thm</u> Two random variables have the same p.m.f./p.d.f. if and only if they have the same moment generating function.

<u>How to Move Between the P.M.F. and the M.G.F. using the Table of M.G.F.'s</u>

It is actually not easy to find m.g.f.'s directly in closed form. Fortunately, the m.g.f.'s of all the random variables we are studying are well known. See the inner cover of the book for a table.

Ex Find the m.g.f. of the binomial random variable with 7 trials and success probability 1/4.

From table, $b(n,p)$ has m.g.f.

$$M(t) = (1-p+pe^t)^n.$$

So $$M(t) = \left(1-\frac{1}{4}+\frac{1}{4}e^t\right)^7$$

$$M(t) = \left(\frac{3}{4}+\frac{1}{4}e^t\right)^7.$$

Ex If the random variable X has m.g.f. $M(t) = \left(\frac{1}{9}+\frac{8}{9}e^t\right)^{100}$, determine its distribution.

From the table, we see that the m.g.f. has the form of the m.g.f. of $b(n,p)$:

$$M(t) = (1-p+pe^t)^n$$

so X is binomial with 100 trials and success probability $\frac{8}{9}$. $f(x) = \frac{100!}{x!(100-x)!}\cdot\left(\frac{8}{9}\right)^x\left(\frac{1}{9}\right)^{100-x}$.

Ex If the random variable X has the the m.g.f. $M(t) = \frac{4e^t}{5-e^t}$, determine its distribution.

From the table, we see that the m.g.f has a form similar to the m.g.f of a geometric random variable:

$$M(t) = \frac{pe^t}{1-(1-p)e^t}$$

$$\frac{4e^t}{5-e^t} = \frac{4e^t}{5-e^t} \cdot \frac{1/5}{1/5} = \frac{\frac{4}{5}e^t}{1-\frac{1}{5}e^t}$$

$\Rightarrow$ X is geometric w/ success probability $p = 4/5$.

$$\Rightarrow \quad f(x) = (1-p)^{x-1} p \qquad x = 1, 2, 3, \ldots$$

$$f(x) = \left(\frac{1}{5}\right)^{x-1} \cdot \left(\frac{4}{5}\right) \qquad x = 1, 2, 3, \ldots$$

Ex If the random variable X has the m.g.f. $M(t) = e^{50(e^t - 1)}$, determine its distribution.

From the table, we see that the m.g.f has a form similar to the m.g.f. of a Poisson random variable:

$$M(t) = e^{\lambda(e^t - 1)}$$

$\Rightarrow$ X is Poisson with mean $\lambda = 50$.

$$\Rightarrow \quad f(x) = \frac{\lambda^x}{x!} \cdot e^{-\lambda} \qquad x = 0, 1, 2, \ldots$$

$$f(x) = \frac{50^x}{x!} e^{-50} \qquad x = 0, 1, 2, \ldots$$

Derivation of Closed Form M.G.F.s of Binomial and Geometric Random Variables

Prop The moment generating function of a $b(n,p)$ random variable is

$$M(t) = (q + pe^t)^n$$

where $q := 1-p$.

Pr

$$M(t) \overset{\text{def}}{=} E(e^{tX}) \overset{\text{Thm}}{=} \sum_x f(x) e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (q + pe^t)^n \quad \text{by the Binomial Thm}$$

$$= (1-p+pe^t)^n$$

$\square$

Prop The moment generating function of a geometric random variable with success probability p is

$$M(t) = \frac{pe^t}{1-qe^t} \quad \text{for } t < -\ln q$$

where $q := 1-p$.

Pr $M(t) \overset{\text{def}}{=} E(e^{tX}) \overset{\text{Thm}}{=} \sum_x f(x) e^{tx}$

$$= \sum_{x=1}^{\infty} q^{x-1} \cdot p \cdot e^{tx}$$

$(e^t)^x = (e^t)^{x-1} \cdot e^t$

$$= \sum_{x=1}^{\infty} q^{x-1} \cdot (pe^t)\, e^{t(x-1)}$$

$$= pe^t \sum_{x=1}^{\infty} q^{x-1} \cdot e^{t(x-1)}$$

$$= pe^t \cdot \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

by geometric series with $|qe^t| < 1$, so $t < -\ln q$.

$$= pe^t \sum_{x=0}^{\infty} (qe^t)^x = \frac{pe^t}{1-qe^t} \quad \square$$

Theorems about Moment Generating Functions

Thm Suppose X is a discrete random variable and suppose its moment generating function exists. Then

$$\left.\frac{d^n M}{dt^n}\right|_{t=0} = E(X^n).$$

In particular, $E(X) = M'(0)$ and $Var(X) = M''(0) - (M'(0))^2$.

Pr

$$M(t) = \sum_x f(x) e^{tx}$$

$$\frac{d^n}{dt^n} M(t) = \frac{d^n}{dt^n} \sum_x f(x) e^{tx}$$

$$= \sum_x f(x) \left(\frac{d}{dt^n} e^{tx} \right)$$

$$= \sum_x f(x) \cdot x^n \cdot e^{tx}$$

Evaluating at $t=0$ gives

$$\left.\frac{d^n m}{dt^n}\right|_{t=0} = \sum_x x^n f(x) = E(X^n).$$

$\square$

Application of Thm:

<u>Cor</u> The expected value of a geometric random variable with success probability p is $\frac{1}{p}$. Its variance is $\frac{1-p}{p^2}$.

<u>Pr</u> $M(t) = \frac{pe^t}{1-qe^t}$ for $t < -\ln q$ from before.

$$\frac{dM}{dt} = \frac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2}$$ by the quotient rule.

$$= \frac{pe^t}{(1-qe^t)^2}$$

Thm $\Rightarrow E(X) = M'(0) = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{p^2} = \frac{1}{p}$

For the variance, do quotient rule again and use $V(X) = M''(0) - (M'(0))^2$. $\square$

Ex (of Thm). If the moment generating function of X is

$$M(t) = \frac{2}{7}e^{3t} + \frac{1}{7}e^{5t} + \frac{4}{7}e^{10t},$$

find the mean, variance, and pmf of X.

Use Thm.

$$M'(t) = \frac{6}{7}e^{3t} + \frac{5}{7}e^{5t} + \frac{40}{7}e^{10t}$$

$$\Rightarrow M'(0) = \frac{6}{7} + \frac{5}{7} + \frac{40}{7} = \frac{51}{7}$$

$$\Rightarrow \boxed{E(X) = \frac{51}{7}} \text{ by Thm.}$$

$$M''(t) = \frac{18}{7}e^{3t} + \frac{25}{7}e^{5t} + \frac{400}{7}e^{10t}$$ by differentiating $M'(t)$.

$$\Rightarrow M''(0) = \frac{18}{7} + \frac{25}{7} + \frac{400}{7} = \frac{443}{7}$$

$$\Rightarrow Var(X) = E(X^2) - E(X)^2 = M''(0) - (M'(0))^2$$

$$\boxed{Var(X) = \frac{443}{7} - \left(\frac{51}{7}\right)^2}$$

The p.m.f. is

X	3	5	10
$f(x)$	$2/7$	$1/7$	$4/7$

Cor Suppose X is a random variable and the m.g.f. of X exists. Then

$$M(t) = \sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n$$

for $t \in (-h, h)$.

Pr Form the Taylor series expansion od $M(t)$ about 0, then use $E(X^n) = M^{(n)}(0)$.

$\square$