2.6 The Poisson Distribution

In this section, we discuss another kind of discrete random variable called Poisson. A Poisson random variable counts the number of occurrences in some continuous interval, assuming the situation satisfies three properties which make it into an approximate Poisson process.

Examples of occurrences counted in a continuous interval are:

- the number of calls to customer service between 9 am and noon, at a local business
- the number of flaws in a 100 ft roll of fabric that is 2 feet wide
- the number of car accidents in Dearborn in a year.

Mathematical Warm-up: Recall the Taylor series expansion for e^{λ} about 0.

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots = e^{\lambda}$$

Then dividing by e^{λ} on both sides gives

$$\sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!} = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \frac{\lambda^4 e^{-\lambda}}{4!} + \cdots = 1$$

Thus, if we define a function

$f : \{0,1,2,\ldots\} \to [0,1]$ by $f(j) = \frac{\lambda^j e^{-\lambda}}{j!}$

where $\lambda > 0$, then f satisfies the properties for a p.m.f.: $f(j) \geq 0$ for all $j \in \operatorname{dom} f$,

$$\sum_{j \in \operatorname{dom} f} f(j) = 1.$$

<u>Def</u> A random variable with image $\mathbb{Z}_{\geq 0}$ has a <u>Poisson distribution</u> if its prob. mass function $f : \mathbb{Z}_{\geq 0} \to [0,1]$ has the form

$$f(j) = \frac{\lambda^j}{j!} e^{-\lambda}$$

for some $\lambda > 0$.

<u>Thm</u> The mean and variance of a Poisson random variable with parameter $\lambda > 0$ are both λ.

When is a physical situation modelled by a Poisson random variable? In the following.

<u>Def</u> Consider counting occurrences in a given continuous time period, region, or length interval. Suppose there exists a positive number $\lambda > 0$ such that:

(a) The numbers of occurrences in non-overlapping intervals are independent.

(b) The probability of exactly one occurrence in a sufficiently short interval of length h is approximately λh:

$$P(\text{one occurrence in int. of length } h) \approx \lambda h.$$

(c) The probability of more than 1 occurrence in a sufficiently short interval h is approximately 0:

$$P(\text{more than 1 occurrence in int. of length } h) \approx 0.$$

If all three requirements are satisfied, then the situation is called an <u>(approximate) Poisson process.</u>

<u>Rem</u> Suppose we have a Poisson process and the average number of occurrences in an interval of length <u>1 unit</u> is λ. Let Y be the random variable which counts the number of occurrences in an interval of length <u>t units.</u> <u>Then</u>

Y is a Poisson random variable with parameter λt and so its p.m.f. is $f: \mathbb{Z}_{\geq 0} \to [0,1]$, $f(j) = \frac{(\lambda t)^j}{j!} e^{-(\lambda t)}$.

<u>Ex</u> Customers arrive at a certain check-out counter at a mean rate of 20 per hour. Suppose the number of arrivals in an hour has a Poisson distribution. Thus $f(j) = \frac{20^j}{j!} e^{-20}$

(i) What is the probability that exactly 30 customers arrive in an hour? $P(Y=30)$ is:

$$f(30) = \frac{20^{30}}{30!} e^{-20} = \text{dpois}(30,20) \approx .0083$$

(ii) What is the probability that exactly 30, 31, or 32 arrive?

$$P(30 \leq Y \leq 32) = f(30) + f(31) + f(32)$$

$$= \frac{20^{30}}{30!} e^{-20} + \frac{20^{31}}{31!} e^{-20} + \frac{20^{32}}{32!} e^{-20}$$

$$= \text{dpois}(30,20) + \text{dpois}(31,20) + \text{dpois}(32,20)$$

$$\approx .01709$$

(iii) What is the probability that at least 15 but no more than 30 arrive?

notice!

$$P(15 \leq Y \leq 30) = F(30) - F(14) = \text{ppois}(30,20) - \text{ppois}(14,20)$$

$$\approx .8817$$

Ex At a particular wood mill, flaws in plywood occur at random with an average of one flaw per 50 square feet. Suppose that the number of flaws per unit area is Poisson distributed. What is the probability that 4 ft × 8 ft sheet of plywood will have at most 1 flaw?

Hint: use $f(j) = \frac{(\lambda t)^j}{j!} e^{-(\lambda t)}$.

Answer: Find the avg number of flaws per sq. foot, call this λ, and let t be the number of sq. feet on one sheet. Then use the formula above.

$$\frac{1 \text{ flaw}}{50 \text{ sq. ft}} = \frac{1 \text{ flaw}}{50 \text{ sq ft}} \cdot \frac{1/50}{1/50} = \frac{1/50 \text{ flaws}}{1 \text{ sq. foot}} \Rightarrow \lambda = \frac{1}{50}$$

One 4 ft × 8 ft sheet has 32 sq. feet.

$$\Rightarrow f(j) = \frac{\left(\frac{1}{50} \cdot 32\right)^j}{j!} e^{-\left(\frac{1}{50} \cdot 32\right)}$$

$$P(Y \le 1) = \frac{\left(\frac{32}{50}\right)^0}{0!} e^{-\frac{32}{50}} + \frac{\left(\frac{32}{50}\right)^1}{1!} e^{-\frac{32}{50}} = (1 + .64)\, e^{-.64}$$

$$= \text{ppois}\left(1, \frac{32}{50}\right) \approx .8648$$

Ex A certain car dealership orders only 5 vehicles of a certain kind per month because there are only 4 people who want to buy that vehicle in a month, on average. Suppose the number of requests follows a Poisson distribution.

(a) What is the expected value of the number sold?

Let $f(j) = \frac{4^j}{j!} e^{-4}$, this is the p.m.f. for the number Y of requests in a given month, (not the p.m.f. for the number sold).

$$\text{expected number sold} = 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) + 4 \cdot f(4) + 5 \cdot \underbrace{\left(1 - \sum_{j=0}^{4} f(j)\right)}_{1 - \text{ppois}(4,4)}$$

$$\approx 3.59$$

(b) How many vehicles of that type should the manager order so that the chance of running out is less than .05?

This means $F(\pi_{.95}) \geq .95$,

$$\pi_{.95} = \text{qpois}(.95, 4) = 8.$$

$$\left(\begin{array}{l} \text{ppois}(8,4) \approx .9786 \\ \text{ppois}(7,4) \approx .9489 \end{array} \right)$$