3.5 The Gamma and Chi-Square Distributions

Earlier, we discussed Poisson processes, and said that a Poisson random variable counts the number of occurrences in an interval of fixed length, assuming a Poisson process. A Poisson random variable is discrete, since it can only take on non-negative integer values.

However, there are also some continuous random variables that are relevant to Poisson processes, for instance the waiting time until the first occurrence, or the second occurrence, or the α-th occurrence. For a Poisson process with mean λ, the waiting time until the α-th occurrence has a gamma distribution density

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta} \qquad 0 \leq x < \infty$$

with parameters α and $\theta := \frac{1}{\lambda}$.

Def A random variable X is said to have a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$ if its density is

$$f(x) = \begin{cases} 0 & \text{for } x \in (0, \infty) \\ \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta} & \text{for } x \in [0, \infty) \end{cases}.$$

Plot!

In R The gamma density is dgamma(x, shape=α, scale=θ).

Thm If a random variable X has a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$, then

$$E(X) = \alpha\theta \quad \text{and} \quad V(X) = \alpha\theta^2.$$

The moment generating function of X is

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}} \quad \text{for } t < \frac{1}{\theta}.$$

Ex Customers enter a mall at a mean rate of $\frac{3}{4}$ per minute, according to a Poisson process. Let X denote the waiting time until the 20th customer arrives.

(a) Find the density of X.

$$f(x) = \frac{1}{19! \cdot \left(\frac{4}{3}\right)^{20}} x^{19} \cdot e^{-x/(4/3)} \quad \text{for } x \in [0, \infty).$$

Because $\alpha = 20$ and $\theta = \frac{1}{\lambda} = \frac{1}{4/3}$,

and $\Gamma(20) = 19!$

(b) Find the mean and variance of X.

$$E(X) = \alpha\theta = 20 \cdot \frac{4}{3} = \frac{80}{3}$$

$$V(X) = \alpha\theta^2 = 20 \cdot \left(\frac{4}{3}\right)^2 = \frac{320}{9}$$

(c) What is the probability the 20th customer arrives after 15 minutes?

$$P(X > 15) = 1 - P(X \leq 15)$$

$$= 1 - \int_0^{15} \frac{1}{19! \left(\frac{4}{3}\right)^{20}} x^{19} \cdot e^{-x/(4/3)} dx$$

$$= 1 - \text{pgamma}(15, \text{shape} = 20, \text{scale} = 4/3)$$

$$\approx .9884$$

Rem Two important special cases of the gamma distribution are:

1) $\alpha = 1 \Rightarrow$ exponential density w/ parameter θ and rate $= 1/\theta$.

2) $\alpha = \frac{r}{2}, \theta = 2, r \in \mathbb{Z}_{>0} \Rightarrow$ chi-square w/ r degrees of freedom

Chi-Square Distributions with r Degrees of Freedom

Let r be a positive integer. Let $\alpha = \frac{r}{2}$ and $\theta = 2$ in the gamma distribution. Then we obtain the chi-square distribution with r degrees of freedom.

Def Let r be a positive integer. The chi-square density with r degrees of freedom is

$$f(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ \dfrac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} x^{\frac{r}{2}-1} \cdot e^{-\frac{x}{2}} & \text{for } x \in [0, \infty) \end{cases}$$

If a random variable X has this density, we say X is $\chi^2(r)$.

$\chi^2(r)$ Distributions in R:

density is dchisq(y, df=r)

c.d.f. is pchisq(y, df=r)

quantile is qchisq(p, df=r).

Higher degree of freedom means more probability is on the right:

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curve(dchisq(y,df=3), xname="y", from=0, to=10)
curve(dchisq(y, df=5), xname="y", from=0 to=10,
                                     add=T)
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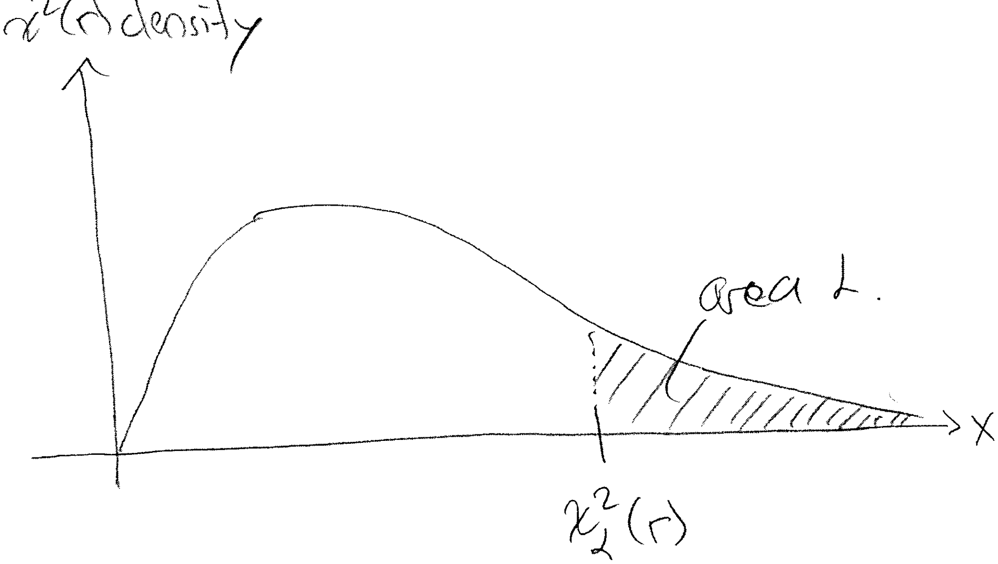
See also figure 3.5-2 on page 152.

<u>Notation</u> If a random variable X has a chi-square density with r degrees of freedom, we say "X is $\chi^2(r)$".

<u>Thm</u> If a random variable X is $\chi^2(r)$, then $E(X) = r$ and $V(X) = 2r$, and its moment generating function is $M(t) = (1-2t)^{-r/2}$ for $t < \frac{1}{2}$.

<u>Notation</u> Let $0 < \alpha < 1$. The <u>upper 100αth percent point for the $\chi^2(r)$ density</u> is denoted with a subscript α as $\chi^2_\alpha(r)$. This is the number $\chi^2_\alpha(r)$ such that the area under the $\chi^2(r)$ density to the right of $\chi^2_\alpha(r)$

is α.

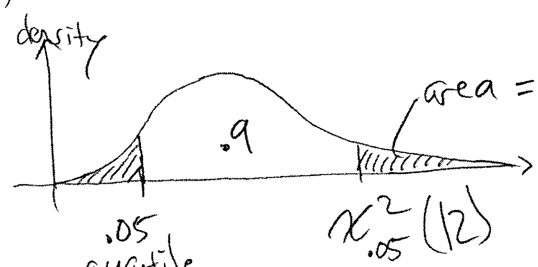


So $P(X \geq \chi^2_\alpha(r)) = \alpha$

In R, we compute $\chi^2_\alpha(r)$ as

$$\chi^2_\alpha(r) = \text{qchisq}(\alpha, \text{df}=r, \text{lower.tail}=F).$$

Ex 3.5 #10 If X is $\chi^2(12)$, find constants a and b such that $P(a < X < b) = .9$ and $P(X < a) = .05$



Find these

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qchisq(.05, df=12, lower.tail=F) ≈ 21.026
qchisq(.05, df=12) ≈ 5.226
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