Definition: (infinite / finite set)

A set B is said to be infinite if there exists a proper subset B_0 of B and a bijection

$$\boxed{f : B \to B_0}$$

Zermelo-Fraenkel (ZF) Axioms

ZF1) 2 sets are identical iff they contain the same elements

Consequence: To show $A = B$, you show $A \subseteq B$ and $B \subseteq A$.

ZF2) Every non-empty set A contains an element x such that

$$x \cap A = \emptyset$$

Consequence: no set is an element of itself.

ZF3) Let A be any set. Let x be a free variable and let $Q(x)$ be a logical formula involving x.

Then

$$B = \{ x \in A \mid Q(x) \}$$

exists (but may be empty)

Example:

$$\{n \in \mathbb{N} : \underbrace{2 \mid n}_{\varphi(n)}\}$$

= even natural numbers

$$\{n \in \mathbb{N} : 2 \mid n \text{ and } 2 \nmid n\}$$

$$= \emptyset$$

ZF4) Let A and B be sets. Then $\exists$ a set C with $A \in C$ and $B \in C$.

Example: $A = \{1, 2\}$

$B = \{3, 4\}$

$C = \{A, B\}$.

ZF5) Let A be a set. Then

$$B = \bigcup_{x \in A} x$$ exists.

Example: $A = \mathbb{N}$

$$B = \bigcup_{n \in \mathbb{N}} n$$

Where n is considered as a <u>set</u>.

ZF6) Let A be a set and let f be a function defined on A. Then if $f(x)$ is a set $\forall x \in A$, then $\exists$ a set B such that $f(A) \subseteq B$.

idea: the range is always contained in a codomain which is a set!

ZF7) $\exists$ an infinite set.

Consequence: $\exists$ a set!

ZFC) Let A be a set and let

$$\mathcal{P}(A) = \{ B \subseteq A \}.$$

i.e. $\mathcal{P}(A)$ is a set whose elements are the subsets of A. Then $\mathcal{P}(A)$ exists.

Example: $A = \{1, 2, 3\}$.

$$P(A) = \{\{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \phi\}$$

$= 8$ elements

$= 2^3$ elements.

If A is finite with n elements, $P(A)$ has 2^n elements.

If we add on additional axiom, we get the Zermelo-Fraenkel-Choice (ZFC) system:

ZFC1) (Axiom of Choice)

Given any collection of sets $\{A_i\}_{i \in I}$, $\exists$ $f: \bigsqcup_{i \in I} A_i \to \bigcup_{i \in I} A_i$ such that f selects exactly one element out of each A_i, $i \in I$.

You can add other axioms - if you like the idea of doing so, consider a career in logic!